CHAPTER

38 PHOTONS AND MATTER WAVES

38-1 What is Physics?

One primary focus of physics is Einstein's theory of relativity, which took us into a world far beyond that of ordinary experience—the world of objects moving at speeds close to the speed of light. Among other surprises, Einstein's theory predicts that the rate at which a clock runs depends on how fast the clock is moving relative to the observer: the faster the motion, the slower the clock rate. This and other predictions of the theory have passed every experimental test devised thus far, and relativity theory has led us to a deeper and more satisfying view of the nature of space and time.

Now you are about to explore a second world that is outside ordinary experience—the subatomic world. You will encounter a new set of surprises that, though they may sometimes seem bizarre, have led physicists step by step to a deeper view of reality.

Quantum physics, as our new subject is called, answers such questions as: Why do the stars shine? Why do the elements exhibit the order that is so apparent in the periodic table? How do transistors and other microelectronic devices work? Why does copper conduct electricity but glass does not? Because quantum physics accounts for all of chemistry, including biochemistry, we need to understand it if we are to understand life itself.

Some of the predictions of quantum physics seem strange even to the physicists and philosophers who study its foundations. Still, experiment after experiment has proved the theory correct, and many have exposed even stranger aspects of the theory. The quantum world is an amusement park full of wonderful rides that are guaranteed to shake up the commonsense world view you have developed since childhood. We begin our exploration of that quantum park with the photon.

38-2 The Photon, the Quantum of Light

Quantum physics (which is also known as *quantum mechanics* and *quantum theory*) is largely the study of the microscopic world. In that world, many quantities are found only in certain minimum (*elementary*) amounts, or integer multiples of those elementary amounts; these quantities are then said to be *quantized*. The elementary amount that is associated with such a quantity is called the **quantum** of that quantity (*quanta* is the plural).

In a loose sense, U.S. currency is quantized because the coin of least value is the penny, or \$0.01 coin, and the values of all other coins and bills are restricted to integer multiples of that least amount. In other words, the currency quantum is \$0.01, and all greater amounts of currency are of the form *n*(\$0.01), where *n* is always a positive integer. For example, you cannot hand someone \$0.755 = 75.5(\$0.01).

In 1905, Einstein proposed that electromagnetic radiation (or simply *light*) is quantized and exists in elementary amounts (quanta) that we now call **photons**. This proposal should seem strange to you because we have just spent several chapters discussing the classical idea that light is a sinusoidal wave, with a wavelength λ, a frequency f, and a speed c such that

$$f = \frac{c}{\lambda}. \tag{38-1}$$

Furthermore, in Chapter 33 we discussed the classical light wave as being an interdependent combination of electric and magnetic fields, each oscillating at frequency f. How can this wave of oscillating fields consist of an elementary amount of something—the light quantum? What *is* a photon?

The concept of a light quantum, or a photon, turns out to be far more subtle and mysterious than Einstein imagined. Indeed, it is still very poorly understood. In this book, we shall discuss only some of the basic aspects of the photon concept, somewhat along the lines of Einstein's proposal.

According to that proposal, the quantum of a light wave of frequency f has the energy

$$E = hf \qquad \text{(photon energy)}. \tag{38-2}$$

Here h is the **Planck constant**, the constant we first met in Eq. 32-23, and which has the value

$$h = 6.63 \times 10^{-34}\ \mathrm{J \cdot s} = 4.14 \times 10^{-15}\ \mathrm{eV \cdot s}. \tag{38-3}$$

The smallest amount of energy a light wave of frequency f can have is hf, the energy of a single photon. If the wave has more energy, its total energy must be an integer multiple of hf, just as the currency in our previous example must be an integer multiple of \$0.01. The light cannot have an energy of, say, $0.6hf$ or $75.5hf$.

Einstein further proposed that when light is absorbed or emitted by an object (matter), the absorption or emission event occurs in the atoms of the object. When light of frequency f is absorbed by an atom, the energy hf of one photon is transferred from the light to the atom. In this *absorption event*, the photon vanishes and the atom is said to absorb it. When light of frequency f is emitted by an atom, an amount of energy hf is transferred from the atom to the light. In this *emission event*, a photon suddenly appears and the atom is said to emit it. Thus, we can have *photon absorption* and *photon emission* by atoms in an object.

For an object consisting of many atoms, there can be many photon absorptions (such as with sunglasses) or photon emissions (such as with lamps). However, each absorption or emission event still involves the transfer of an amount of energy equal to that of a single photon of the light.

When we discussed the absorption or emission of light in previous chapters, our examples involved so much light that we had no need of quantum physics, and we got by with classical physics. However, in the late 20th century, technology became advanced enough that single-photon experiments could be conducted and put to practical use. Since then quantum physics has become part of standard engineering practice, especially in optical engineering.

CHECKPOINT 1

Rank the following radiations according to their associated photon energies, greatest first: (a) yellow light from a sodium vapor lamp, (b) a gamma ray emitted by a radioactive nucleus, (c) a radio wave emitted by the antenna of a commercial radio station, (d) a microwave beam emitted by airport traffic control radar.

Answer:

b, a, d, c

Emission and absorption of light as photons

A sodium vapor lamp is placed at the center of a large sphere that absorbs all the light reaching it. The rate at which the lamp emits energy is 100 W; assume that the emission is entirely at a wavelength of 590 nm. At what rate are photons absorbed by the sphere?

KEY IDEAS

The light is emitted and absorbed as photons. We assume that all the light emitted by the lamp reaches (and thus is absorbed by) the sphere. So, the rate R at which photons are absorbed by the sphere is equal to the rate R_{emit} at which photons are emitted by the lamp.

Calculations:

That rate is

$$R_{\text{emit}} = \frac{\text{rate of energy emission}}{\text{energy per emitted photon}} = \frac{P_{\text{emit}}}{E}.$$

Into this we can substitute from Eq. 38-2 ($E = hf$), Einstein's proposal about the energy E of each quantum of light (which we here call a photon in modern language). We can then write the absorption rate as

$$R = R_{\text{emit}} = \frac{P_{\text{emit}}}{hf}.$$

Using Eq. 38-1 ($f = c/\lambda$) to substitute for f and then entering known data, we obtain

$$\begin{aligned} R &= \frac{P_{\text{emit}}\lambda}{hf} \\ &= \frac{(100\text{W})(590 \times 10^{-9}\text{m})}{(6.6. \times 10^{-34}\text{J}\cdot\text{s})(2.998 \times 10^{8}\text{m/s})} \qquad \text{(Answer)} \\ &= 2.97 \times 10^{20}\text{photons/s.} \end{aligned}$$

38-3 The Photoelectric Effect

If you direct a beam of light of short enough wavelength onto a clean metal surface, the light will cause electrons to leave that surface (the light will *eject* the electrons from the surface). This **photoelectric effect** is used in many devices, including TV cameras, camcorders, and night vision viewers. Einstein supported his photon concept by using it to explain this effect, which simply cannot be understood without quantum physics.

Let us analyze two basic photoelectric experiments, each using the apparatus of Fig. 38-1, in which light of frequency f is directed onto target T and ejects electrons from it. A potential difference V is maintained between target T and collector cup C to sweep up these electrons, said to be **photoelectrons**. This collection produces a **photoelectric current** i that is measured with meter A.

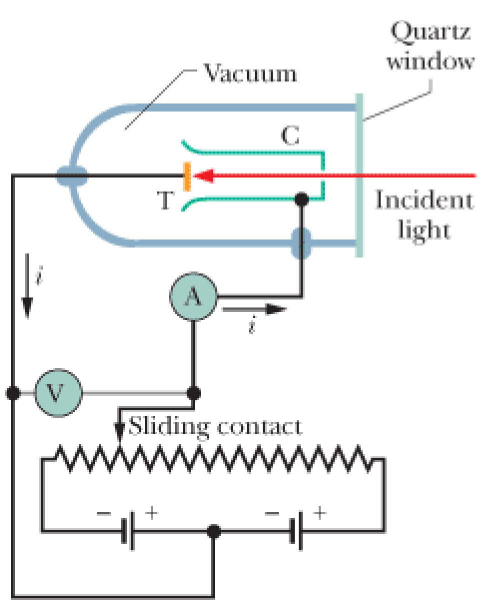


Figure 38-1 An apparatus used to study the photoelectric effect. The incident light shines on target T, ejecting electrons, which are collected by collector cup C. The electrons move in the circuit in a direction opposite the conventional current arrows. The batteries and the variable resistor are used to produce and adjust the electric potential difference between T and C.

First Photoelectric Experiment

We adjust the potential difference V by moving the sliding contact in Fig. 38-1 so that collector C is slightly negative with respect to target T. This potential difference acts to slow down the ejected electrons. We then vary V until it reaches a certain value, called the **stopping potential** V_{stop}, at which point the reading of meter A has just dropped to zero. When $V = V_{stop}$, the most energetic ejected electrons are turned back just before reaching the collector. Then K_{max}, the kinetic energy of these most energetic electrons, is

$$K_{max} = eV_{stop}, \tag{38-4}$$

where e is the elementary charge.

Measurements show that for light of a given frequency, K_{max} *does not depend on the intensity of the light source*. Whether the source is dazzling bright or so feeble that you can scarcely detect it (or has some intermediate brightness), the maximum kinetic energy of the ejected electrons always has the same value.

This experimental result is a puzzle for classical physics. Classically, the incident light is a sinusoidally oscillating electromagnetic wave. An electron in the target should oscillate sinusoidally due to the oscillating electric force on it from the wave's electric field. If the amplitude of the electron's oscillation is great enough, the electron should break free of the target's surface—that is, be ejected from the target. Thus, if we increase the amplitude of the wave and its oscillating electric field, the electron should get a more energetic “kick” as it is being ejected. *However, that is not what happens*. For a given frequency, intense light beams and feeble light beams give exactly the same maximum kick to ejected electrons.

The actual result follows naturally if we think in terms of photons. Now the energy that can be transferred from the incident light to an electron in the target is that of a single photon. Increasing the light intensity increases the *number* of photons in the light, but the photon energy, given by Eq. 38-2 ($E = hf$), is unchanged because the frequency is unchanged. Thus, the energy transferred to the kinetic energy of an electron is also unchanged.

Second Photoelectric Experiment

Now we vary the frequency f of the incident light and measure the associated stopping potential V_{stop}. Figure 38-2 is a plot of V_{stop} versus f. Note that the photoelectric effect does not occur if the frequency is below a certain **cutoff frequency** f_0 or, equivalently, if the wavelength is greater than the corresponding **cutoff wavelength** $\lambda_0 = c/f_0$. This is so *no matter how intense the incident light is*.

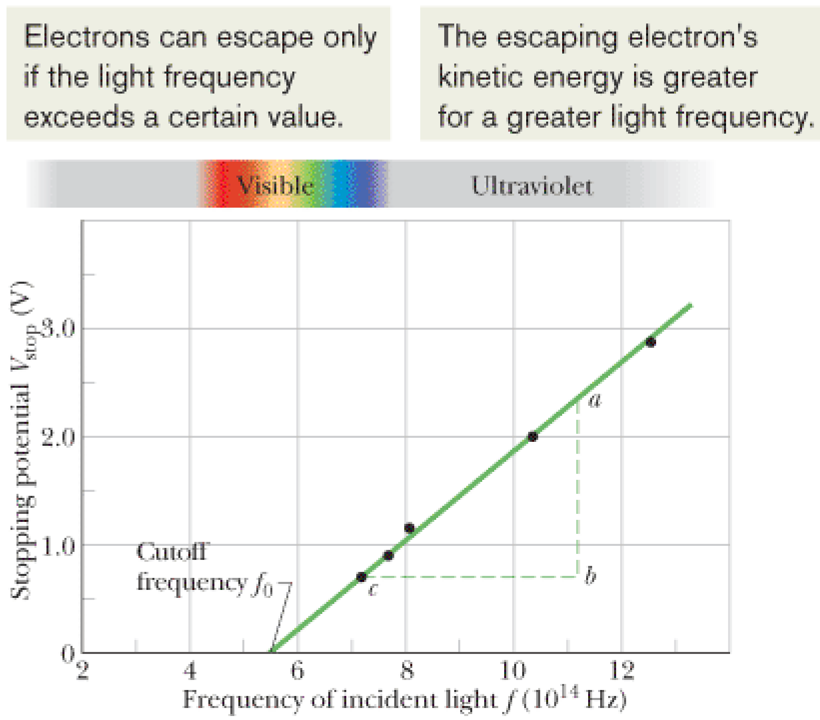


Figure 38-2 The stopping potential V_{stop} as a function of the frequency/of the incident light for a sodium target T in the apparatus of Fig. 38-1.
(Data reported by R. A. Millikan in 1916.)

This is another puzzle for classical physics. If you view light as an electromagnetic wave, you must expect that no matter how low the frequency, electrons can always be ejected by light if you supply them with enough energy—that is, if you use a light source that is bright enough. *That is not what happens*. For light below the cutoff frequency f_0, the photoelectric effect does not occur, no matter how bright the light source.

The existence of a cutoff frequency is, however, just what we should expect if the energy is transferred via photons. The electrons within the target are held there by electric forces. (If they weren't, they would drip out of the target due to the gravitational force on them.) To just escape from the target, an electron must pick up a certain minimum energy Φ, where Φ is a property of the target material called its **work function**. If the energy hf transferred to an electron by a photon exceeds the work function of the material (if $hf > \Phi$), the electron can escape the target. If the energy transferred does not exceed the work function (that is, if $hf < \Phi$), the electron cannot escape. This is what Fig. 38-2 shows.

The Photoelectric Equation

Einstein summed up the results of such photoelectric experiments in the equation

$$hf = K_{max} + \Phi \qquad \text{(photoelectric equation)}. \tag{38-5}$$

This is a statement of the conservation of energy for a single photon absorption by a target with work function Φ. Energy equal to the photon's energy hf is transferred to a single electron in the material of the target. If the electron is to escape from the target, it must pick up energy at least equal to Φ. Any additional energy ($hf - \Phi$) that the electron acquires from the photon appears as kinetic energy K of the electron. In the most favorable circumstance, the electron can escape through the surface without losing any of this kinetic energy in the process; it then appears outside the target with the maximum possible kinetic energy K_{max}.

Let us rewrite Eq. 38-5 by substituting for K_{max} from Eq. 38-4 ($K_{max} = eV_{stop}$).

After a little rearranging we get

$$V_{stop} = \left(\frac{h}{e}\right)f - \frac{\Phi}{e}. \tag{38-6}$$

The ratios h/e and Φ/e are constants, and so we would expect a plot of the measured stopping potential V_{stop} versus the frequency f of the light to be a straight line, as it is in Fig. 38-2. Further, the slope of that straight line should be h/e. As a check, we measure ab and bc in Fig. 38-2 and write

$$\begin{aligned}\frac{h}{e} &= \frac{ab}{bc} = \frac{2.35\,\text{V} - 0.72\,\text{V}}{(11.2 \times 10^{14} - 7.2 \times 10^{14})\,\text{Hz}} \\ &= 4.1 \times 10^{-15}\,\text{V} \cdot \text{s}.\end{aligned}$$

Multiplying this result by the elementary charge e, we find

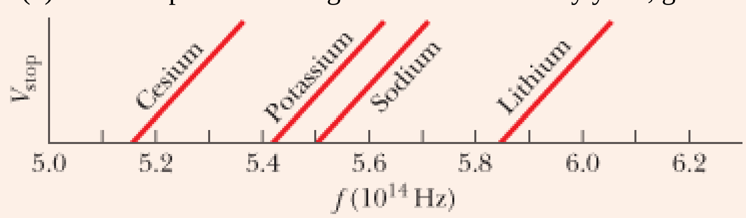
$$h = (4.1 \times 10^{-15}\,\text{V} \cdot \text{s})(1.6 \times 10^{-19}\,\text{C}) = 6.6 \times 10^{-34}\,\text{J} \cdot \text{s},$$

which agrees with values measured by many other methods.

An aside: An explanation of the photoelectric effect certainly requires quantum physics. For many years, Einstein's explanation was also a compelling argument for the existence of photons. However, in 1969 an alternative explanation for the effect was found that used quantum physics but did not need the concept of photons. Light *is* in fact quantized as photons, but Einstein's explanation of the photoelectric effect is not the best argument for that fact.

CHECKPOINT 2

The figure shows data like those of Fig. 38-2 for targets of cesium, potassium, sodium, and lithium. The plots are parallel. (a) Rank the targets according to their work functions, greatest first. (b) Rank the plots according to the value of h they yield, greatest first.



Answer:

(a) lithium, sodium, potassium, cesium; (b) all tie

Photoelectric effect and work function

Find the work function Φ of sodium from Fig. 38-2.

KEY IDEAS

We can find the work function Φ from the cutoff frequency f (which we can measure on the plot). The reasoning is this: At the cutoff frequency, the kinetic energy K_{max} in Eq. 38-5 is zero. Thus, all the energy hf that is transferred from a photon to an electron goes into the electron's escape, which requires an energy of Φ.

Calculations:

From that last idea, Eq. 38-5 then gives us, with $f = f_0$,

$$hf_0 = 0 + \Phi = \Phi.$$

In Fig. 38-2, the cutoff frequency f_0 is the frequency at which the plotted line intercepts the horizontal frequency axis, about 5.5×10^{14} Hz. We then have

$$\begin{aligned} \Phi &= hf_0 = (6.63 \times 10^{-34}\,\mathrm{J \cdot s})(5.5 \times 10^{14}\,\mathrm{Hz}) \\ &= 3.6 \times 10^{-19}\,\mathrm{J} = 2.3\,\mathrm{eV}. \end{aligned} \quad \text{(Answer)}$$

38-4 Photons Have Momentum

In 1916, Einstein extended his concept of light quanta (photons) by proposing that a quantum of light has linear momentum. For a photon with energy hf, the magnitude of that momentum is

$$p = \frac{hf}{c} = \frac{h}{\lambda} \quad \text{(photon momentum)}, \tag{38-7}$$

where we have substituted for f from Eq. 38-1 ($f = c/\lambda$). Thus, when a photon interacts with matter, energy *and* momentum are transferred, *as if* there were a collision between the photon and matter in the classical sense (as in Chapter 9).

In 1923, Arthur Compton at Washington University in St. Louis carried out an experiment that supported the view that both momentum and energy are transferred via photons. He arranged for a beam of x rays of wavelength λ to be directed onto a target made of carbon, as shown in Fig. 38-3. An x ray is a form of electromagnetic radiation, at high frequency and thus small wavelength. Compton measured the wavelengths and intensities of the x rays that were scattered in various directions from his carbon target.

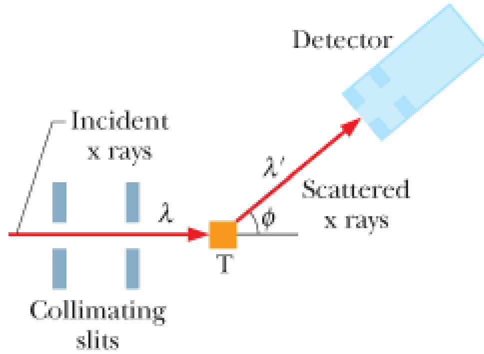


Figure 38-3 Compton's apparatus. A beam of x rays of wavelength λ = 71.1 pm is directed onto a carbon target T. The x rays scattered from the target are observed at various angles to the direction of the incident beam. The detector measures both the intensity of the scattered x rays and their wavelength.

Figure 38-4 shows his results. Although there is only a single wavelength (λ = 71.1 pm) in the incident x-ray beam, we see that the scattered x rays contain a range of wavelengths with two prominent intensity peaks. One peak is centered about the incident wavelength λ, the other about a wavelength λ' that is longer than λ by an amount $\Delta\lambda$, which is called the **Compton shift**. The value of the Compton shift varies with the angle at which the scattered x rays are detected and is greater for a greater angle.

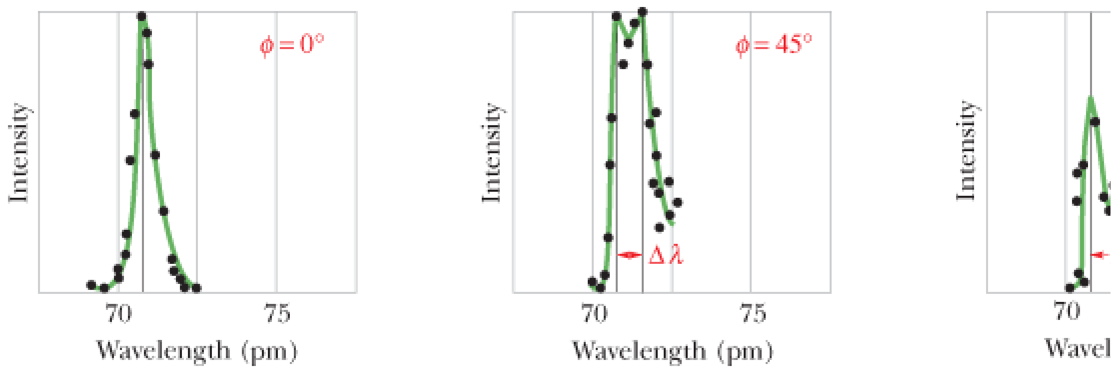


Figure 38-4 Compton's results for four values of the scattering angle . Note that the Compton shift $\Delta\lambda$ increases as the scattering angle increases.

Figure 38-4 is still another puzzle for classical physics. Classically, the incident x-ray beam is a sinusoidally oscillating electromagnetic wave. An electron in the carbon target should oscillate sinusoidally due to the oscillating electric force on it from the wave's electric field. Further, the electron should oscillate at the same frequency as the wave and should send out waves *at this same frequency*, as if it were a tiny transmitting antenna. Thus, the x rays scattered by the electron should have the same frequency, and the same wavelength, as the x rays in the incident beam—but they don't.

Compton interpreted the scattering of x rays from carbon in terms of energy and momentum transfers, via photons, between the incident x-ray beam and loosely bound electrons in the carbon target. Let us see, first conceptually and then quantitatively, how this quantum physics interpretation leads to an understanding of Compton's results.

Suppose a single photon (of energy $E = hf$) is associated with the interaction between the incident x-ray beam and a stationary electron. In general, the direction of travel of the x ray will change (the x ray is scattered), and the electron will recoil, which means that the electron has obtained some kinetic energy. Energy is conserved in this isolated interaction. Thus, the energy of the scattered photon ($E' = hf'$) must be less than that of the incident photon. The scattered x rays must then have a lower frequency f' and thus a longer wavelength λ' than the incident x rays, just as Compton's experimental results in Fig. 38-4 show.

For the quantitative part, we first apply the law of conservation of energy. Figure 38-5 suggests a “collision” between an x ray and an initially stationary free electron in the target. As a result of the collision, an x ray of wavelength λ' moves off at an angle and the electron moves off at an angle θ, as shown. Conservation of energy then gives us

$$hf = hf' + K,$$

in which hf is the energy of the incident x-ray photon, hf' is the energy of the scattered x-ray photon, and K is the kinetic energy of the recoiling electron. Because the electron may recoil with a speed comparable to that of light, we must use the relativistic expression of Eq. 37-52,

$$K = mc^2(\gamma - 1),$$

for the electron's kinetic energy. Here m is the electron's mass and γ is the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}}.$$

Compton scattering

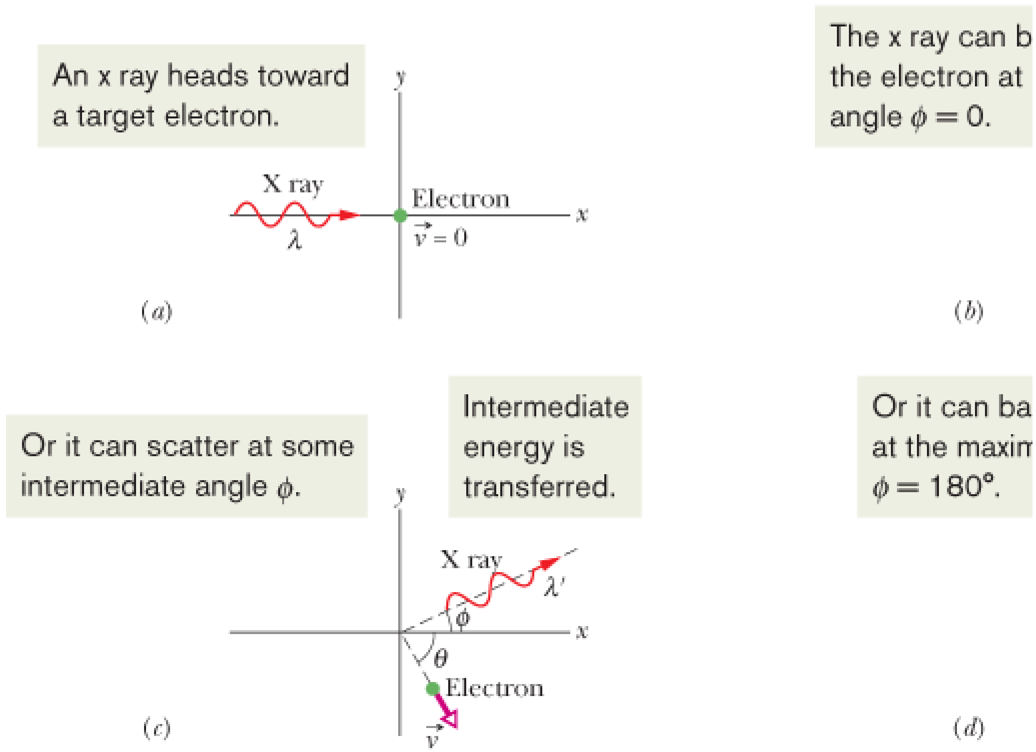


Figure 38-5(*a*) An x ray approaches a stationary electron. The x ray can (*b*) bypass the electron (forward scatter) with no energy or momentum transfer, (*c*) scatter at some intermediate angle with an intermediate energy and momentum transfer, or (*d*) backscatter with the maximum energy and momentum transfer.

Substituting for K in the conservation of energy equation yields

$$hf = hf' + mc^2(\gamma - 1).$$

Substituting c/λ for f and c/λ' for f' then leads to the new energy conservation equation

$$\frac{h}{\lambda} = \frac{h}{\lambda'} + mc(\gamma - 1). \qquad (38\text{-}8)$$

Next we apply the law of conservation of momentum to the x-ray–electron collision of Fig. 38-5. From Eq. 38-7 ($p = h/\lambda$), the magnitude of the momentum of the incident photon is h/λ, and that of the scattered photon is h/λ'. From Eq. 37-41, the magnitude for the recoiling electron's momentum is $p = \gamma mv$. Because we have a two-dimensional situation, we write separate equations for the conservation of momentum along the x and y axes, obtaining

$$\frac{h}{\lambda} = \frac{h}{\lambda'} \cos\varphi + \gamma mv \cos\theta \quad (x \text{ axis}) \qquad (38\text{-}9)$$

and

$$0 = \frac{h}{\lambda'} \sin\varphi - \gamma mv \sin\theta \quad (y \text{ axis}). \qquad (38\text{-}10)$$

We want to find $\Delta\lambda$ (= λ' - λ), the Compton shift of the scattered x rays. Of the five collision variables (λ, λ', v, , and θ) that appear in Eqs. 38-8, 38-9, and 38-10, we choose to eliminate v and θ, which deal only with the recoiling electron. Carrying out the algebra (it is somewhat complicated) leads to

$$\Delta\lambda = \frac{h}{mc}(1 - \cos\phi) \quad \text{(Compton shift).} \tag{38-11}$$

Equation 38-11 agrees exactly with Compton's experimental results.

The quantity h/mc in Eq. 38-11 is a constant called the **Compton wavelength**. Its value depends on the mass m of the particle from which the x rays scatter. Here that particle is a loosely bound electron, and thus we would substitute the mass of an electron for m to evaluate the *Compton wavelength for Compton scattering from an electron*.

CHECKPOINT 3

Compare Compton scattering for x rays ($\lambda \approx 20$ pm) and visible light ($\lambda \approx 500$ nm) at a particular angle of scattering. Which has the greater (a) Compton shift, (b) fractional wavelength shift, (c) fractional energy loss, and (d) energy imparted to the electron?

Answer:

(a) same;(b) - (d) x rays

A Loose End

The peak at the incident wavelength λ (= 71.1 pm) in Fig. 38-4 still needs to be explained. This peak arises not from interactions between x rays and the very loosely bound electrons in the target but from interactions between x rays and the electrons that are *tightly* bound to the carbon atoms making up the target. Effectively, each of these latter collisions occurs between an incident x ray and an entire carbon atom. If we substitute for m in Eq. 38-11 the mass of a carbon atom (which is about 22 000 times that of an electron), we see that $\Delta\lambda$ becomes about 22 000 times smaller than the Compton shift for an electron—too small to detect. Thus, the x rays scattered in these collisions have the same wavelength as the incident x rays.

Compton scattering of light by electrons

X rays of wavelength λ = 22 pm (photon energy = 56 keV) are scattered from a carbon target, and the scattered rays are detected at 85° to the incident beam.

(a) What is the Compton shift of the scattered rays?

KEY IDEA

The Compton shift is the wavelength change of the x rays due to scattering from loosely bound electrons in a target. Further, that shift depends on the angle at which the scattered x rays are detected, according to Eq. 38-11. The shift is zero for forward scattering at angle = 0°, and it is maximum for back scattering at angle = 180°. Here we have an intermediate situation at angle = 85°.

Calculation:

Substituting 85° for that angle and 9.11×10^{-31}kg for the electron mass (because the scattering is from electrons) in Eq. 38-11 gives us

$$\begin{aligned}\Delta\lambda &= \frac{h}{mc}(1-\cos\phi)\\ &= \frac{\left(6.63\times10^{-34}\,\mathrm{J\cdot s}\right)(1-\cos 85°)}{\left(9.11\times10^{-31}\,\mathrm{kg}\right)\left(3.00\times10^{8}\,\mathrm{m/s}\right)} \qquad \text{(Answer)}\\ &= 2.21\times10^{-12}\,\mathrm{m}\approx 2.2\,\mathrm{pm}.\end{aligned}$$

(b) What percentage of the initial x-ray photon energy is transferred to an electron in such scattering?

KEY IDEA

We need to find the *fractional energy loss* (let us call it *frac*) for photons that scatter from the electrons:

$$frac = \frac{\text{energy loss}}{\text{initial energy}} = \frac{E-E'}{E}.$$

Calculations:

From Eq. 38-2 ($E = hf$), we can substitute for the initial energy E and the detected energy E' of the x rays in terms of frequencies. Then, from Eq. 38-1 ($f = c/\lambda$), we can substitute for those frequencies in terms of the wavelengths. We find

$$\begin{aligned}frac &= \frac{hf-hf'}{hf} = \frac{c\lambda - c\lambda'}{c/\lambda} = \frac{\lambda'-\lambda}{\lambda'}\\ &= \frac{\Delta\lambda}{\lambda+\Delta\lambda}.\end{aligned} \qquad (38\text{-}12)$$

Substitution of data yields 2.21 pm

$$frac = \frac{2.21\,\mathrm{pm}}{22\,\mathrm{pm}+2.21\,\mathrm{pm}} = 0.091, \text{ or } 9.0\,\%. \qquad \text{(Answer)}$$

Although the Compton shift $\Delta\lambda$ is independent of the wavelength λ of the incident x rays (see Eq. 38-11), the *fractional* photon energy loss of the x rays does depend on λ, increasing as the wavelength of the incident radiation decreases, as indicated by Eq. 38-12.

38-5 Light as a Probability Wave

A fundamental mystery in physics is how light can be a wave (which spreads out over a region) in classical physics but be emitted and absorbed as photons (which originate and vanish at points) in quantum physics. The double-slit experiment of Section 35-4 lies at the heart of this mystery. Let us discuss three versions of that experiment.

The Standard Version

Figure 38-6 is a sketch of the original experiment carried out by Thomas Young in 1801 (see also Fig. 35-8). Light shines on screen *B*, which contains two narrow parallel slits. The light waves emerging from the two slits spread out by diffraction and overlap on screen *C* where, by interference, they form a pattern of alternating intensity maxima and minima. In Section 35-4 we took the existence of these interference fringes as compelling evidence for the wave nature of light.

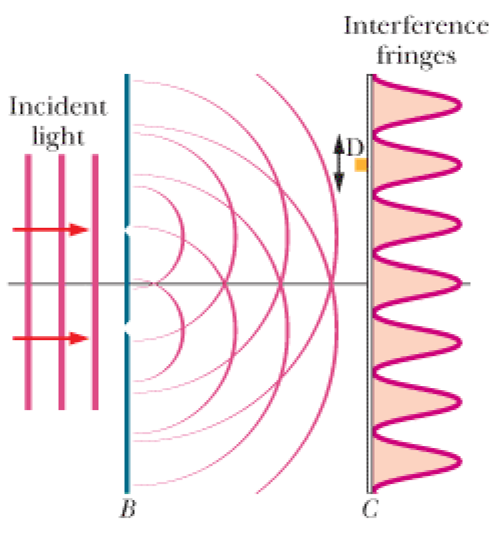


Figure 38-6 Light is directed onto screen *B*, which contains two parallel slits. Light emerging from these slits spreads out by diffraction. The two diffracted waves overlap at screen *C* and form a pattern of interference fringes. A small photon detector D in the plane of screen *C* generates a sharp click for each photon that it absorbs.

Let us place a tiny photon detector D at one point in the plane of screen *C*. Let the detector be a photoelectric device that clicks when it absorbs a photon. We would find that the detector produces a series of clicks, randomly spaced in time, each click signaling the transfer of energy from the light wave to the screen via a photon absorption. If we moved the detector very slowly up or down as indicated by the black arrow in Fig. 38-6, we would find that the click rate increases and decreases, passing through alternate maxima and minima that correspond exactly to the maxima and minima of the interference fringes.

The point of this thought experiment is as follows. We cannot predict when a photon will be detected at any particular point on screen *C*; photons are detected at individual points at random times. We can, however, predict that the relative *probability* that a single photon will be detected at a particular point in a specified time interval is proportional to the light intensity at that point.

We know from Eq. 33-26 $\left(I = E_{rms}^2 / c\mu_0\right)$ in Section 33-5 that the intensity I of a light wave at any point is proportional to the square of E_m, the amplitude of the oscillating electric field vector of the wave at that point. Thus,

> The probability (per unit time interval) that a photon will be detected in any small volume centered on a given point in a light wave is proportional to the square of the amplitude of the wave's electric field vector at that point.

We now have a probabilistic description of a light wave, hence another way to view light. It is not only an electromagnetic wave but also a **probability wave**. That is, to every point in a light wave we can attach a numerical probability (per unit time interval) that a photon can be detected in any small volume centered on that point.

The Single-Photon Version

A single-photon version of the double-slit experiment was first carried out by G. I. Taylor in 1909 and has been repeated many times since. It differs from the standard version in that the light source in the Taylor experiment is so extremely feeble that it emits only one photon at a time, at random intervals. Astonishingly, interference fringes still build up on screen *C* if the experiment runs long enough (several months for Taylor's early experiment).

What explanation can we offer for the result of this single-photon double-slit experiment? Before we can even consider the result, we are compelled to ask questions like these: If the photons move through the apparatus one at a time, through which of the two slits in screen *B* does a given photon pass? How does a given photon even "know" that there is another slit present so that interference is a possibility? Can a single photon somehow pass through both slits and interfere with itself?

Bear in mind that the only thing we can know about photons is when light interacts with matter—we have no way of detecting them without an interaction with matter, such as with a detector or a screen. Thus, in the experiment of Fig. 38-6, all we can know is that photons originate at the light source and vanish at the screen. Between source and screen, we cannot know what the photon is or does. However, because an interference pattern eventually builds up on the screen, we can speculate that each photon travels from source to screen *as a wave* that fills up the space between source and screen and then vanishes in a photon absorption at some point on the screen, with a transfer of energy and momentum to the screen at that point.

We *cannot* predict where this transfer will occur (where a photon will be detected) for any given photon originating at the source. However, we *can* predict the probability that a transfer will occur at any given point on the screen. Transfers will tend to occur (and thus photons will tend to be absorbed) in the regions of the bright fringes in the interference pattern that builds up on the screen. Transfers will tend *not* to occur (and thus photons will tend *not* to be absorbed) in the regions of the dark fringes in the built-up pattern. Thus, we can say that the wave traveling from the source to the screen is a *probability wave*, which produces a pattern of "probability fringes" on the screen.

The Single-Photon, Wide-Angle Version

In the past, physicists tried to explain the single-photon double-slit experiment in terms of small packets of classical light waves that are individually sent toward the slits. They would define these small packets as photons. However, modern experiments invalidate this explanation and definition. Figure 38-7 shows the arrangement of one of these experiments, reported in 1992 by Ming Lai and Jean-Claude Diels of the University of New Mexico. Source S contains molecules that emit photons at well separated times. Mirrors M.$_1$ and M_2 are positioned to reflect light that the source emits along two distinct paths, 1 and 2, that are separated by an angle θ, which is close to 180°. This arrangement differs from the standard two-slit experiment, in which the angle between the paths of the light reaching two slits is very small.

A *single* photon can take widely different paths and still interfere with itself.

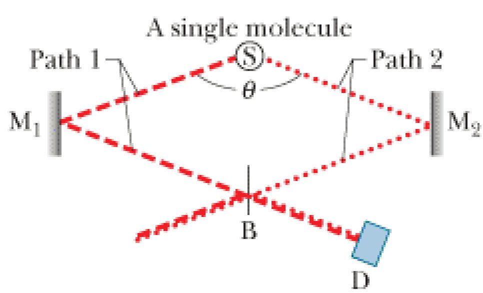


Figure 38-7 The light from a single photon emission in source S travels over two widely separated paths and interferes with itself at detector D after being recombined by beam splitter B.
(After Ming Lai and Jean-Claude Diels, Journal of the Optical Society of America B, ***9****, 2290–2294, December 1992.)*

After reflection from mirrors M_1 and M_2, the light waves traveling along paths 1 and 2 meet at beam splitter B. (A beam splitter is an optical device that transmits half the light incident upon it and reflects the other half.) On the right side of the beam splitter in Fig. 38-7, the light wave traveling along path 2 and reflected by B combines with the light wave traveling along path 1 and transmitted by B. These two waves then interfere with each other as they arrive at detector D (a *photomultiplier tube* that can detect individual photons).

The output of the detector is a randomly spaced series of electronic pulses, one for each detected photon. In the experiment, the beam splitter is moved slowly in a horizontal direction (in the reported experiment, a distance of only about 50 μu, m maximum), and the detector output is recorded on a chart recorder. Moving the beam splitter changes the lengths of paths 1 and 2, producing a phase shift between the light waves arriving at detector D. Interference maxima and minima appear in the detector's output signal.

This experiment is difficult to understand in traditional terms. For example, when a molecule in the source emits a single photon, does that photon travel along path 1 or path 2 in Fig. 38-7 (or along any other path)? Or can it move in both directions at once? To answer, we assume that when a molecule emits a photon, a probability wave radiates in all directions from it. The experiment samples this wave in two of those directions, chosen to be nearly opposite each other.

We see that we can interpret all three versions of the double-slit experiment if we assume that (1) light is generated in the source as photons, (2) light is absorbed in the detector as photons, and (3) light travels between source and detector as a probability wave.

38-6 Electrons and Matter Waves

In 1924, French physicist Louis de Broglie made the following appeal to symmetry: A beam of light is a wave, but it transfers energy and momentum to matter only at points, via photons. Why can't a beam of particles have the same properties? That is, why can't we think of a moving electron—or any other particle—as a **matter wave** that transfers energy and momentum to other matter at points?

In particular, de Broglie suggested that Eq. 38-7 ($p = h/\lambda$) might apply not only to photons but also to electrons. We used that equation in Section 38-4 to assign a momentum p to a photon of light with wavelength λ. We now use it, in the form

$$\lambda = \frac{h}{p} \qquad \text{(de Broglie wavelength)}, \tag{38-13}$$

to assign a wavelength λ to a particle with momentum of magnitude p. The wavelength calculated from Eq. 38-13 is called the **de Broglie wavelength** of the moving particle. De Broglie's prediction of the existence of matter waves was first verified experimentally in 1927, by C. J. Davisson and L. H. Germer of the Bell Telephone Laboratories and by George P. Thomson of the University of Aberdeen in Scotland.

Figure 38-8 shows photographic proof of matter waves in a more recent experiment. In the experiment, an interference pattern was built up when electrons were sent, *one by one*, through a double-slit apparatus. The apparatus was like the ones we have previously used to demonstrate optical interference, except that the viewing screen was similar to an old-fashioned television screen. When an electron hit the screen, it caused a flash of light whose position was recorded.

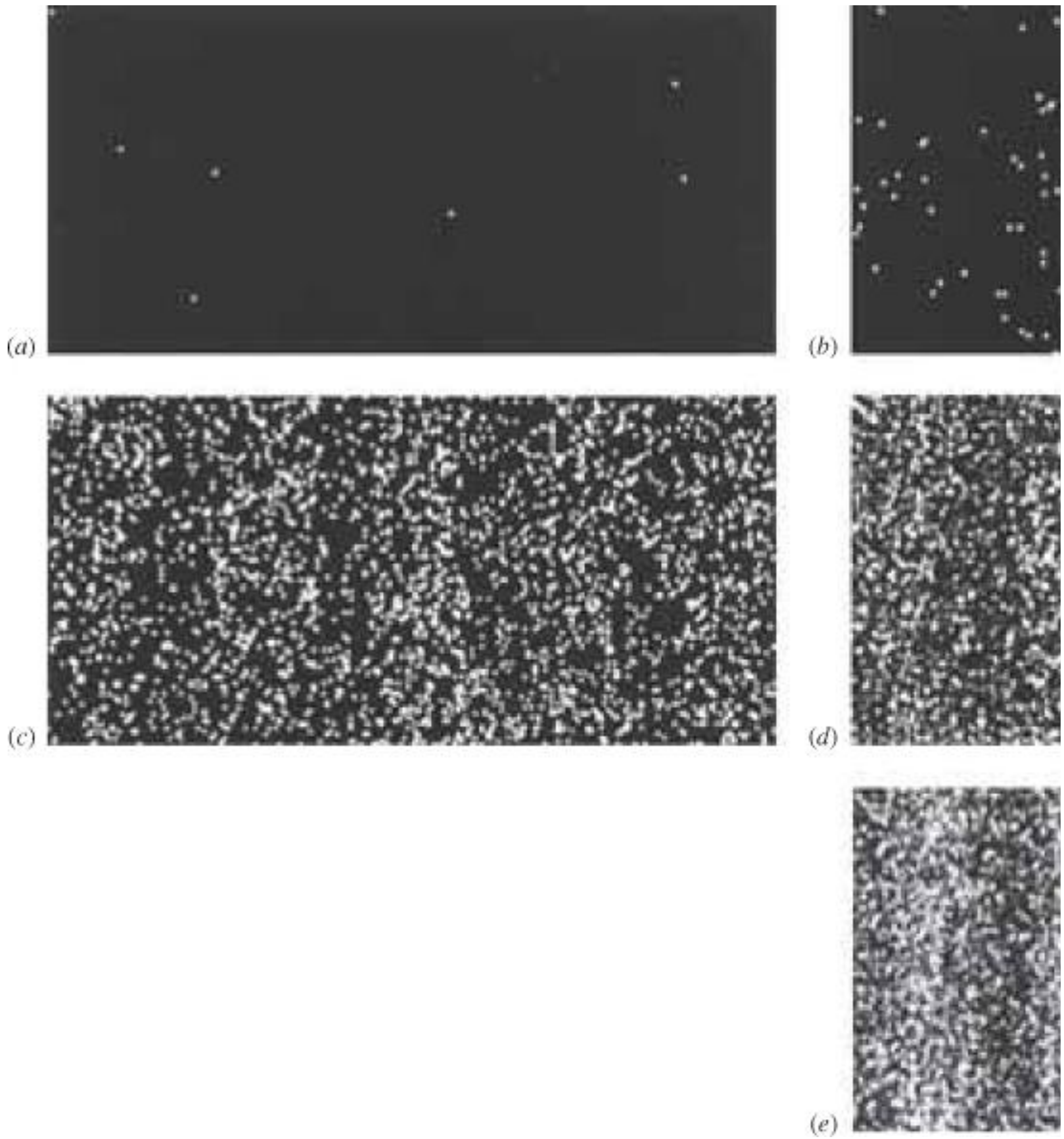


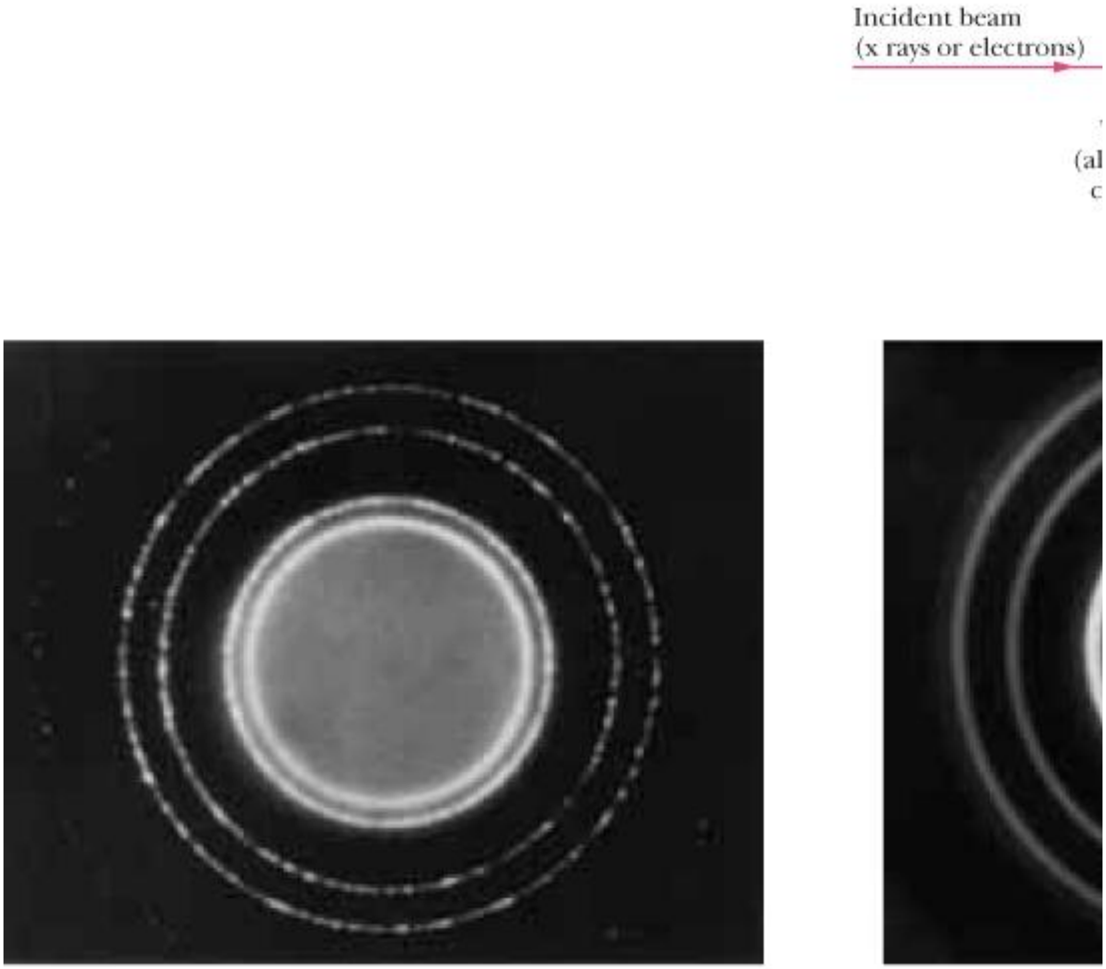
Figure 38-8 Photographs showing the buildup of an interference pattern by a beam of electrons in a two-slit interference experiment like that of Fig. 38-6. Matter waves, like light waves, are *probability waves*. The approximate numbers of electrons involved are (*a*) 7, (*b*) 100, (*c*) 3000, (*d*) 20 000, and (*e*) 70 000.
(Courtesy A. Tonomura, J. Endo, T. Matsuda, and T. Kawasaki/Advanced Research Laboratory, Hitachi, Ltd., Kokubinju, Tokyo, H. Ezawa, Department of Physics, Gakushuin University, Mejiro, Tokyo)

The first several electrons (top two photos) revealed nothing interesting and seemingly hit the screen at random points. However, after many thousands of electrons were sent through the apparatus, a pattern appeared on the screen, revealing fringes where many electrons had hit the screen and fringes where few had hit the screen. The pattern is exactly what we would expect for wave interference. Thus, *each* electron passed through the apparatus as a matter wave—the portion of the matter wave that traveled through one slit interfered with the portion that traveled through the other slit. That interference then determined the probability that the electron would materialize at a given point on the screen, hitting the screen there. Many electrons materialized in regions corresponding to bright fringes in optical interference, and few electrons materialized in regions corresponding to dark fringes.

Similar interference has been demonstrated with protons, neutrons, and various atoms. In 1994, it was demonstrated with iodine molecules I_2, which are not only 500 000 times more massive than electrons but far more complex. In 1999, it was demonstrated with the even more complex *fullerenes* (or *buckyballs*) C_{60} and C_{70}. (Fullerenes are molecules of carbon atoms that are arranged in a structure resembling a soccer ball, 60 carbon atoms in C_{60} and 70 carbon atoms in C_{70}.) Apparently, such small objects as electrons, protons, atoms, and molecules travel as matter waves. However, as we consider larger and more complex objects, there must come a point at which we are no longer justified in considering the wave nature of an object. At that point, we are back in our familiar nonquantum world, with the physics of earlier chapters of this book. In short, an electron is a matter wave and can undergo interference with itself, but a cat is not a matter wave and cannot undergo interference with itself (which must be a relief to cats).

The wave nature of particles and atoms is now taken for granted in many scientific and engineering fields. For example, electron diffraction and neutron diffraction are used to study the atomic structures of solids and liquids, and electron diffraction is used to study the atomic features of surfaces on solids.

Figure 38-9*a* shows an arrangement that can be used to demonstrate the scattering of either x rays or electrons by crystals. A beam of one or the other is directed onto a target consisting of a layer of tiny aluminum crystals. The x rays have a certain wavelength λ. The electrons are given enough energy so that their de Broglie wavelength is the same wavelength λ. The scatter of x rays or electrons by the crystals produces a circular interference pattern on a photographic film. Figure 38-9*b* shows the pattern for the scatter of x rays, and Fig. 38-9*c* shows the pattern for the scatter of electrons. The patterns are the same—both x rays and electrons are waves.



(*b*)

Figure 38-9(*a*) An experimental arrangement used to demonstrate, by diffraction techniques, the wave-like character of the incident beam. Photographs of the diffraction patterns when the incident beam is (*b*) an x-ray beam (light wave) and (*c*) an electron beam (matter wave). Note that the two patterns are geometrically identical to each other. *(Photo (b) Cameca, Inc. Photo (c) from PSSC film "Matter Waves," courtesy Education Development Center, Newton, Massachusetts)*

Waves and Particles

Figures 38-8 and 38-9 are convincing evidence of the *wave* nature of matter, but we have countless experiments that suggest its *particle* nature. Figure 38-10, for example, shows the tracks of particles (rather than waves) revealed in a bubble chamber. When a charged particle passes through the liquid hydrogen that fills such a chamber, the particle causes the liquid to vaporize along the particle's path. A series of bubbles thus marks the path, which is usually curved due to a magnetic field set up perpendicular to the plane of the chamber.

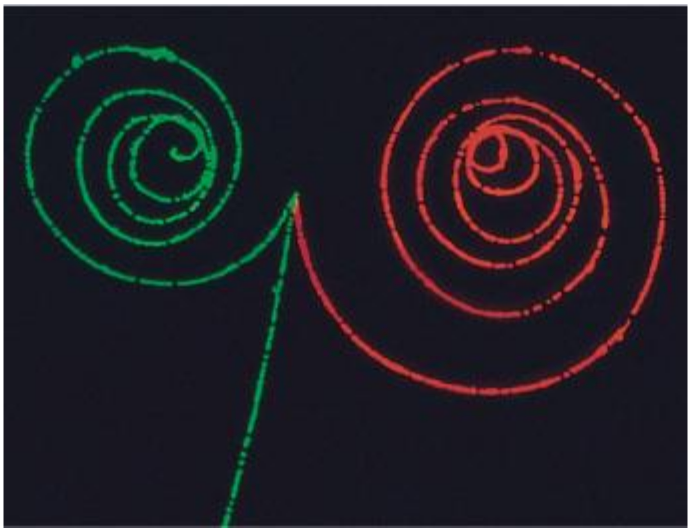


Figure 38-10 A bubble-chamber image showing where two electrons (paths color coded green) and one positron (red) moved after a gamma ray entered the chamber.
(Lawrence Berkeley Laboratory/Science Photo Library/Photo Researchers)

In Fig. 38-10, a gamma ray left no track when it entered at the top because the ray is electrically neutral and thus caused no vapor bubbles as it passed through the liquid hydrogen. However, it collided with one of the hydrogen atoms, kicking an electron out of that atom; the curved path taken by the electron to the bottom of the photograph has been color coded green. Simultaneous with the collision, the gamma ray transformed into an electron and a positron in a pair production event (see Eq. 21-15). Those two particles then moved in tight spirals (color coded green for the electron and red for the positron) as they gradually lost energy in repeated collisions with hydrogen atoms. Surely these tracks are evidence of the particle nature of the electron and positron, but is there any evidence of waves in Fig. 38-10?

To simplify the situation, let us turn off the magnetic field so that the strings of bubbles will be straight. We can view each bubble as a detection point for the electron. Matter waves traveling between detection points such as I and F in Fig. 38-11 will explore all possible paths, a few of which are shown.

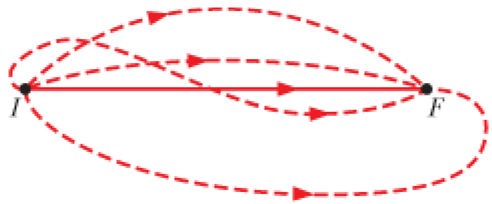


Figure 38-11 A few of the many paths that connect two particle detection points I and F. Only matter waves that follow paths close to the straight line between these points interfere constructively. For all other paths, the waves following any pair of neighboring paths interfere destructively. Thus, a matter wave leaves a straight track.

In general, for every path connecting I and F (except the straight-line path), there will be a neighboring path such that matter waves following the two paths cancel each other by interference. This is not true, however, for the straight-line path joining I and F; in this case, matter waves traversing all neighboring paths reinforce the wave following the direct path. You can think of the bubbles that form the track as a series of detection points at which the matter wave undergoes constructive interference.

CHECKPOINT 4

For an electron and a proton that have the same (a) kinetic energy, (b) momentum, or (c) speed, which particle has the shorter de Broglie wavelength?

Answer:

(a) proton; (b) same; (c) proton

de Broglie wavelength of an electron

What is the de Broglie wavelength of an electron with a kinetic energy of 120 eV?

KEY IDEAS

(1) We can find the electron's de Broglie wavelength λ from Eq. 38-13 ($\lambda = h/p$) if we first find the magnitude of its momentum p. (2) We find p from the given kinetic energy K of the electron. That kinetic energy is much less than the rest energy of an electron (0.511 MeV, from Table 37-3). Thus, we can get by with the classical approximations for momentum p (= mv) and kinetic energy K (= $\frac{1}{2}mv^2$).

Calculations:

We are given the value of the kinetic energy. So, in order to use the de Broglie relation, we first solve the kinetic energy equation for v and then substitute into the momentum equation, finding

$$\begin{aligned} p &= \sqrt{2mK} \\ &= \sqrt{(2)(9.11\times10^{-31}\,\text{kg})(120\,\text{eV})(1.60\times10^{-19}\,\text{J/eV})} \\ &= 5.91\times10^{-24}\,\text{kg}\cdot\text{m/s}. \end{aligned}$$

From Eq. 38-13 then

$$\begin{aligned}\lambda &= \frac{h}{p}\\ &= \frac{6.63\times10^{-34}\,\mathrm{J\cdot s}}{5.91\times10^{-24}\,\mathrm{kg\cdot m/s}} \qquad \text{(Answer)}\\ &= 1.12\times10^{-10}\,\mathrm{m} = 112\,\mathrm{pm}.\end{aligned}$$

This wavelength associated with the electron is about the size of a typical atom. If we increase the electron's kinetic energy, the wavelength becomes even smaller.

38-7 Schrödinger's Equation

A simple traveling wave of any kind, be it a wave on a string, a sound wave, or a light wave, is described in terms of some quantity that varies in a wave-like fashion. For light waves, for example, this quantity is $\vec{E}(x, y, z, t)$, the electric field component of the wave. Its observed value at any point depends on the location of that point and on the time at which the observation is made.

What varying quantity should we use to describe a matter wave? We should expect this quantity, which we call the **wave function** $\Psi(x, y, z, t)$, to be more complicated than the corresponding quantity for a light wave because a matter wave, in addition to energy and momentum, transports mass and (often) electric charge. It turns out that Ψ, the uppercase Greek letter psi, usually represents a function that is complex in the mathematical sense; that is, we can always write its values in the form $a + ib$, in which a and b are real numbers and $i^2 = -1$.

In all the situations you will meet here, the space and time variables can be grouped separately and ψ can be written in the form

$$\Psi(x, y, z, t) = \psi(x, y, z)e^{-i\omega t}, \qquad (38\text{-}14)$$

where ω ($= 2\pi f$) is the angular frequency of the matter wave. Note that ψ, the lowercase Greek letter psi, represents only the space-dependent part of the complete, time-dependent wave function Ψ. We shall focus on ψ. Two questions arise: What is meant by the wave function? How do we find it?

What does the wave function mean? It has to do with the fact that a matter wave, like a light wave, is a probability wave. Suppose that a matter wave reaches a particle detector that is small; then the probability that a particle will be detected in a specified time interval is proportional to $|\psi|^2$, where $|\psi|$ is the absolute value of the wave function at the location of the detector. Although ψ is usually a complex quantity, $|\psi|^2$ is always both real and positive. It is, then, $|\psi|^2$, which we call the **probability density**, and not ψ, that has *physical* meaning. Speaking loosely, the meaning is this:

The probability (per unit time) of detecting a particle in a small volume centered on a given point in a matter wave is proportional to the value of $|\psi|^2$ at that point.

Because ψ is usually a complex quantity, we find the square of its absolute value by multiplying ψ by ψ^*, the *complex conjugate* of ψ. (To find ψ^* we replace the imaginary number *i* in ψ with *-i*, wherever it occurs.)

How do we find the wave function? Sound waves and waves on strings are described by the equations of Newtonian mechanics. Light waves are described by Maxwell's equations. Matter waves are described by **Schrödinger's equation**, advanced in 1926 by Austrian physicist Erwin Schrödinger.

Many of the situations that we shall discuss involve a particle traveling in the *x* direction through a region in which forces acting on the particle cause it to have a potential energy $U(x)$. In this special case, Schrödinger's equation reduces to

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}[E - U(x)]\psi = 0 \quad \text{(Schrödinger's equation, one-dimensional motion),} \tag{38-15}$$

in which E is the total mechanical energy of the moving particle. (We do *not* consider mass energy in this nonrelativistic equation.) We cannot derive Schrödinger's equation from more basic principles; it *is* the basic principle.

If $U(x)$ in Eq. 38-15 is zero, that equation describes a **free particle**—that is, a moving particle on which no net force acts. The particle's total energy in this case is all kinetic, and thus E in Eq. 38-15 is $\frac{1}{2}mv^2$. That equation then becomes

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}\left(\frac{mv^2}{2}\right)\psi = 0,$$

which we can recast as

$$\frac{d^2\psi}{dx^2} + \left(2\pi\frac{p}{h}\right)^2\psi = 0.$$

To obtain this equation, we replaced mv with the momentum p and regrouped terms.

From Eq. 38-13 ($\lambda = h/p$) we recognize p/h in the equation above as $1/\lambda$, where λ is the de Broglie wavelength of the moving particle. We further recognize $2\pi/\lambda$ as the *angular wave number k*, which we defined in Eq. 16-5. With these substitutions, the equation above becomes

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \quad \text{(Schrödinger's equation, free particle).} \tag{38-16}$$

The most general solution of Eq. 38-16 is

$$\psi(x) = Ae^{ikx} + Be^{-ikx}, \tag{38-17}$$

in which A and B are arbitrary constants. You can show that this equation is indeed a solution of Eq. 38-16 by substituting $\psi(x)$ and its second derivative into that equation and noting that an identity results.

If we combine Eqs. 38-14 and 38-17, we find, for the time-dependent wave function Ψ of a free particle traveling in the *x* direction,

$$\begin{aligned}\Psi(x,t) &= \psi(x)e^{-i\omega t} = \left(Ae^{ikx} + Be^{-ikx}\right)e^{-i\omega t} \\ &= Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}.\end{aligned} \tag{38-18}$$

Finding the Probability Density $|\psi|^2$

In Section 16-5 we saw that *any function F* of the form *F(kx* ± *ωt*) represents a traveling wave. This applies to exponential functions like those in Eq. 38-18 as well as to the sinusoidal functions we have used to describe waves on strings. For a general angle *θ*, these two representations of functions are related by

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta.$$

The first term on the right in Eq. 38-18 thus represents a wave traveling in the positive direction of *x* and the second term represents a wave traveling in the negative direction of *x*. However, we have assumed that the free particle we are considering travels only in the positive direction of *x*. To reduce the general solution (Eq. 38-18) to our case of interest, we choose the arbitrary constant *B* in Eqs. 38-18 and 38-17 to be zero. At the same time, we relabel the constant *A* as ψ_0. Equation 38-17 then becomes

$$\psi(x) = \psi_0 e^{ikx}. \tag{38-19}$$

To calculate the probability density, we take the square of the absolute value:

$$|\psi|^2 = \left|\psi_0 e^{ikx}\right|^2 = \left(\psi_0^2\right)\left|e^{ikx}\right|^2.$$

Now, because

$$\left|e^{ikx}\right|^2 = \left(e^{ikx}\right)\left(e^{ikx}\right)^* = e^{ikx}e^{-ikx} = e^{ikx-ikx} = e^0 = 1,$$

we get

$$|\psi|^2 = \left(\psi_0^2\right)(1)^2 = \psi_0^2 \quad \text{(a constant)}.$$

Figure 38-12 is a plot of the probability density $|\psi|^2$ versus *x* for a free particle—a straight line parallel to the *x* axis from - ∞ to +∞. We see that the probability density $|\psi|^2$ is the same for all values of *x*, which means that the particle has equal probabilities of being *anywhere* along the *x* axis. There is no distinguishing feature by which we can predict a most likely position for the particle. That is, all positions are equally likely. We'll see what this means in the next section.

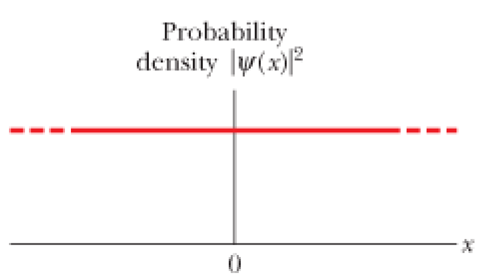


Figure 38-12 A plot of the probability density $|\psi|^2$ for a free particle moving in the positive x direction. Since $|\psi|^2$ has the same constant value for all values of x, the particle has the same probability of detection at all points along its path.

38-8 Heisenberg's Uncertainty Principle

Our inability to predict the position of a free particle, as indicated by Fig. 38-12, is our first example of **Heisenberg's uncertainty principle**, proposed in 1927 by German physicist Werner Heisenberg. It states that measured values cannot be assigned to the position $\vec{r}$ and the momentum $\vec{p}$ of a particle simultaneously with unlimited precision.

In terms of $\hbar = h/2\pi$ (called "h-bar"), the principle tells us

$$\begin{aligned} \Delta x \cdot \Delta p_x &\geq \hbar \\ \Delta y \cdot \Delta p_y &\geq \hbar \qquad \text{(Heisenberg's uncertainty principle).} \\ \Delta z \cdot \Delta p_z &\geq \hbar \end{aligned} \tag{38-20}$$

Here Δx and Δp_x represent the intrinsic uncertainties in the measurements of the x components of $\vec{r}$ and $\vec{p}$, with parallel meanings for the y and z terms. Even with the best measuring instruments, each product of a position uncertainty and a momentum uncertainty in Eq. 38-20 will be greater than $\hbar$, *never* less.

The particle whose probability density is plotted in Fig. 38-12 is a free particle; that is, no force acts on it, and so its momentum $\vec{p}$ must be constant. We implied—without making a point of it—that we can determine $\vec{p}$ with absolute precision; in other words, we assumed that $\Delta p_x = \Delta p_y = \Delta p_z = 0$ in Eq. 38-20. That assumption then requires $\Delta x \to \infty$, $\Delta y \to \infty$, and $\Delta z \to \infty$. With such infinitely great uncertainties, the position of the particle is completely unspecified.

Do not think that the particle *really has* a sharply defined position that is, for some reason, hidden from us. If its momentum can be specified with absolute precision, the words "position of the particle" simply lose all meaning. The particle in Fig. 38-12 can be found *with equal probability* anywhere along the x axis.

Uncertainty principle: position and momentum

Assume that an electron is moving along an x axis and that you measure its speed to be 2.05×10^6 m/s, which can be known with a precision of 0.50%. What is the minimum uncertainty (as allowed by the uncertainty principle in quantum theory) with which you can simultaneously measure the position of the electron along the x axis?

KEY IDEA

The minimum uncertainty allowed by quantum theory is given by Heisenberg's uncertainty principle in Eq. 38-20. We need only consider components along the *x* axis because we have motion only along that axis and want the uncertainty Δx in location along that axis. Since we want the minimum allowed uncertainty, we use the equality instead of the inequality in the *x*-axis part of Eq. 38-20, writing $\Delta x \cdot \Delta p_x = h/2\pi$.

Calculations:

To evaluate the uncertainty Δp_x in the momentum, we must first evaluate the momentum component p_x. Because the electron's speed v_x is much less than the speed of light *c*, we can evaluate p_x with the classical expression for momentum instead of using a relativistic expression. We find

$$\begin{aligned} p_x &= mv_x = \left(9.11 \times 10^{-31}\,\mathrm{kg}\right)\left(2.05 \times 10^{-6}\,\mathrm{ms}\right) \\ &= 1.87 \times 10^{-24}\,\mathrm{kg \cdot m/s.} \end{aligned}$$

The uncertainty in the speed is given as 0.50% of the measured speed. Because p_x depends directly on speed, the uncertainty Δp_x in the momentum must be 0.50% of the momentum:

$$\begin{aligned} \Delta p_x &= (0.0050)\,p_x \\ &= \left(0.0050\right)\left(1.87 \times 10^{-24}\,\mathrm{kg \cdot m/s}\right) \\ &= 9.35 \times 10^{-27}\,\mathrm{kg \cdot m/s.} \end{aligned}$$

Then the uncertainty principle gives us

$$\begin{aligned} \Delta x = \frac{h/2\pi}{\Delta p_x} &= \frac{\left(6.63 \times 10^{-34}\,\mathrm{J \cdot s}\right)/2\pi}{9.35 \times 10^{-27}\,\mathrm{kg \cdot m/s}} \\ &= 1.13 \times 10^{-8}\,\mathrm{m} = 11\,\mathrm{nm}, \end{aligned} \qquad \text{(Answer)}$$

which is about 100 atomic diameters. Given your measurement of the electron's speed, it makes no sense to try to pin down the electron's position to any greater precision.

38-9 Barrier Tunneling

Suppose you slide a puck over frictionless ice toward an ice-covered hill (Fig. 38-13). As the puck climbs the hill, kinetic energy *K* is transformed into gravitational potential energy *U*. If the puck reaches the top, its potential energy is U_b. Thus, the puck can pass over the top only if its initial mechanical energy $E > U_b$. Otherwise, the puck eventually stops its climb up the left side of the hill and slides back to the left. For instance, if U_b = 20 J and E = 10 J, you cannot expect the puck to pass over the hill. We say that the hill acts as a **potential energy barrier** (or, for short, a **potential barrier**) and that, in this case, the barrier has a *height* of U_b = 20 J.

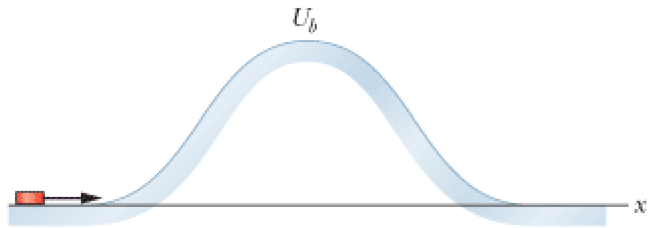


Figure 38-13 A puck slides over frictionless ice toward a hill. The puck's gravitational potential at the top of the hill will be U_b.

Figure 38-14 shows a potential barrier for a nonrelativistic electron traveling along an idealized wire of negligible thickness. The electron, with mechanical energy E, approaches a region (the barrier) in which the electric potential V_b is negative. Because it is negatively charged, the electron will have a positive potential energy U_b (= qV_b) in that region (Fig. 38-15). If $E > U_b$, we expect the electron to pass through the barrier region and come out to the right of $x = L$ in Fig. 38-14. Nothing surprising there. If $E < U_b$, we expect the electron to be unable to pass through the barrier region. Instead, it should end up traveling leftward, much as the puck would slide back down the hill in Fig. 38-13 if the puck has $E < U_b$.

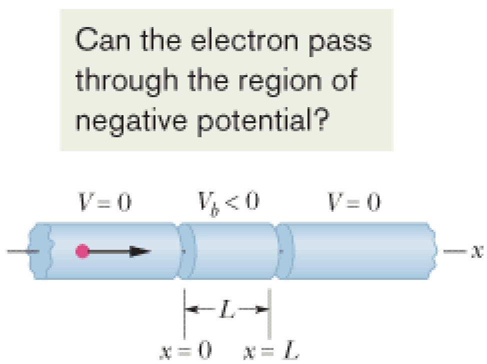


Figure 38-14 The elements of an idealized thin wire in which an electron (the dot) approaches a negative electric potential V_b in the region $x = 0$ to $x = L$.

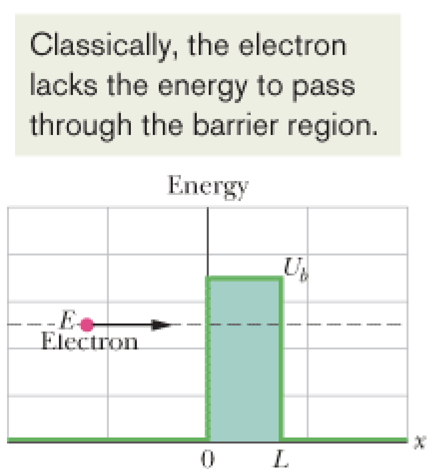


Figure 38-15 An energy diagram containing two plots for the situation of Fig. 38-13: (1) The electron's mechanical energy E is plotted when the electron is at any coordinate $x < 0$. (2) The electron's electric potential energy U is plotted as a function of the electron's position x, *assuming* that the electron can reach any value of x. The nonzero part of the plot (the potential barrier) has height U_b and thickness L.

However, something astounding can happen to the electron when $E < U_b$. Because it is a matter wave, the electron has a finite probability of leaking (or, better, *tunneling*) through the barrier and materializing on the other side, moving rightward with energy E as though nothing (strange or otherwise) had happened in the region of $0 \le x \le L$.

The wave function $\psi(x)$ describing the electron can be found by solving Schrödinger's equation (Eq. 38-15) separately for the three regions in Fig. 38-14: (1) to the left of the barrier, (2) within the barrier, and (3) to the right of the barrier. The arbitrary constants that appear in the solutions can then be chosen so that the values of $\psi(x)$ and its derivative with respect to x join smoothly (no jumps, no kinks) at $x = 0$ and at $x = L$. Squaring the absolute value of $\psi(x)$ then yields the probability density.

Figure 38-16 shows a plot of the result. The oscillating curve to the left of the barrier (for $x < 0$) is a combination of the incident matter wave and the reflected matter wave (which has a smaller amplitude than the incident wave). The oscillations occur because these two waves, traveling in opposite directions, interfere with each other, setting up a standing wave pattern.

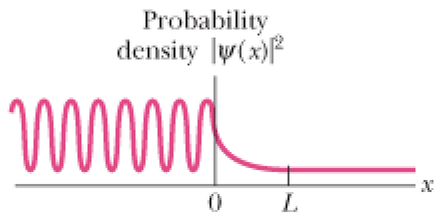


Figure 38-16 A plot of the probability density $|\psi|^2$ of the electron matter wave for the situation of Fig. 38-15. The value of $|\psi|^2$ is nonzero to the right of the potential barrier.

Within the barrier (for $0 < x < L$) the probability density decreases exponentially with x. However, if L is small, the probability density is not quite zero at $x = L$.

To the right of the barrier (for $x > L$), the probability density plot describes a transmitted (through the barrier) wave with low but constant amplitude. Thus, the electron can be detected in this region but with a relatively small probability. (Compare this part of the figure with Fig. 38-12 for a free particle.)

We can assign a **transmission coefficient** T to the incident matter wave and the barrier. This coefficient gives the probability with which an approaching electron will be transmitted through the barrier—that is, that tunneling will occur. As an example, if $T = 0.020$, then of every 1000 electrons fired at the barrier, 20 (on average) will tunnel through it and 980 will be reflected. The transmission coefficient $T$is approximately

$$T \approx e^{-2bL}, \tag{38-21}$$

in which

$$b = \sqrt{\frac{8\pi^2 m(U_b - E)}{h^2}}, \tag{38-22}$$

and e is the exponential function. Because of the exponential form of Eq. 38-21, the value of T is very sensitive to the three variables on which it depends: particle mass m, barrier thickness L, and energy difference U_b - E. (Because we do not include relativistic effects here, E does not include mass energy.)

Barrier tunneling finds many applications in technology, including the tunnel diode, in which a flow of electrons produced by tunneling can be rapidly turned on or off by controlling the barrier height. The 1973 Nobel Prize in physics was shared by three "tunnelers," Leo Esaki (for tunneling in semiconductors), Ivar Giaever (for tunneling in superconductors), and Brian Josephson (for the Josephson junction, a rapid quantum switching device based on tunneling). The 1986 Nobel Prize was awarded to Gerd Binnig and Heinrich Rohrer for development of the scanning tunneling microscope.

CHECKPOINT 5

Is the wavelength of the transmitted wave in Fig. 38-16 larger than, smaller than, or the same as that of the incident wave?

Answer:

same

The Scanning Tunneling Microscope (STM)

The size of details that can be seen in an optical microscope is limited by the wavelength of the light the microscope uses (about 300 nm for ultraviolet light). The size of details that can be seen in the image that opens this chapter is far smaller and thus requires much smaller wavelengths. The waves used are electron matter waves, but they do not scatter from the surface being examined the way waves do in an optical microscope. Instead, the images we see are created by electrons tunneling through potential barriers at the tip of a *scanning tunneling microscope* (STM).

Figure 38-17 shows the heart of the scanning tunneling microscope. A fine metallic tip, mounted at the intersection of three mutually perpendicular quartz rods, is placed close to the surface to be examined. A small potential difference, perhaps only 10 mV, is applied between tip and surface.

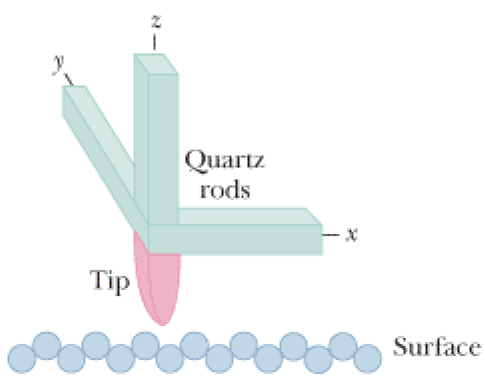


Figure 38-17 The essence of a scanning tunneling microscope (STM). Three quartz rods are used to scan a sharply pointed conducting tip across the surface of interest and to maintain

a constant separation between tip and surface. The tip thus moves up and down to match the contours of the surface, and a record of its movement provides information for a computer to create an image of the surface.

Crystalline quartz has an interesting property called *piezoelectricity:* When an electric potential difference is applied across a sample of crystalline quartz, the dimensions of the sample change slightly. This property is used to change the length of each of the three rods in Fig. 38-17, smoothly and by tiny amounts, so that the tip can be scanned back and forth over the surface (in the x and y directions) and also lowered or raised with respect to the surface (in the z direction).

The space between the surface and the tip forms a potential energy barrier, much like that plotted in Fig. 38-15. If the tip is close enough to the surface, electrons from the sample can tunnel through this barrier from the surface to the tip, forming a tunneling current.

In operation, an electronic feedback arrangement adjusts the vertical position of the tip to keep the tunneling current constant as the tip is scanned over the surface. This means that the tip-surface separation also remains constant during the scan. The output of the device is a video display of the varying vertical position of the tip, hence of the surface contour, as a function of the tip position in the *xy* plane.

An STM not only can provide an image of a static surface, it can also be used to manipulate atoms and molecules on a surface, such as was done in forming the *quantum corral* shown in Fig. 39-12 in the next chapter. In a process known as lateral manipulation, the STM probe is initially brought down near a molecule, close enough that the molecule is attracted to the probe without actually touching it. The probe is then moved across the background surface (such as platinum), dragging the molecule with it until the molecule is in the desired location. Then the probe is backed up away from the molecule, weakening and then eliminating the attractive force on the molecule. Although the work requires very fine control, a design can eventually be formed. In Fig. 39-12, an STM probe has been used to move 48 iron atoms across a copper surface and into a circular corral 14 nm in diameter, in which electrons can be trapped.

Barrier tunneling by matter wave

Suppose that the electron in Fig. 38-15, having a total energy E of 5.1 eV, approaches a barrier of height U_b = 6.8 eV and thickness L = 750 pm.

(a) What is the approximate probability that the electron will be transmitted through the barrier, to appear (and be detectable) on the other side of the barrier?

KEY IDEA

The probability we seek is the transmission coefficient T as given by Eq. 38-21 ($T \approx e^{-2bL}$), where

$$b = \sqrt{\frac{8\pi^2 m(U_b - E)}{h^2}}.$$

Calculations:

The numerator of the fraction under the square-root sign is

$$\left(8\pi^2\right)\left(9.11\times10^{-31}\text{kg}\right)(6.8\text{eV}-5.1\text{eV})\times\left(1.60\times10^{-19}\text{J / eV}\right)=1.956\times10^{-47}$$

Thus,

$$b=\sqrt{\frac{\left(1.956\times10^{-47}\text{J}\cdot\text{kg}\right)}{\left(6.63\times10^{-34}\text{J}\cdot\text{kg}\right)^2}}=6.67\times10^{9}\text{m}^{-1}.$$

The (dimensionless) quantity $2bL$ is then

$$2bL=(2)\left(6.67\times10^{9}\text{m}^{-1}\right)\left(750\times10^{-12}\text{m}\right)=10.0$$

and, from Eq. 38-21, the transmission coefficient is

$$T\approx e^{-2bL}=e^{-10.0}=45\times10^{-6}. \qquad \text{(Answer)}$$

Thus, of every million electrons that strike the barrier, about 45 will tunnel through it, each appearing on the other side with its original total energy of 5.1 eV. (The transmission through the barrier does not alter an electron's energy or any other property.)

(b) What is the approximate probability that a proton with the same total energy of 5.1 eV will be transmitted through the barrier, to appear (and be detectable) on the other side of the barrier?

Reasoning: The transmission coefficient T (and thus the probability of transmission) depends on the mass of the particle. Indeed, because mass m is one of the factors in the exponent of e in the equation for T, the probability of transmission is very sensitive to the mass of the particle. This time, the mass is that of a proton (1.67×10^{-27} kg), which is significantly greater than that of the electron in (a). By substituting the proton's mass for the mass in (a) and then continuing as we did there, we find that $T \approx 10^{-186}$. Thus, although the probability that the proton will be transmitted is not exactly zero, it is barely more than zero. For even more massive particles with the same total energy of 5.1 eV, the probability of transmission is exponentially lower.

sec. 38-2 The Photon, the Quantum of Light

•1 Monochromatic light (that is, light of a single wavelength) is to be absorbed by a sheet of photographic film and thus recorded on the film. Photon absorption will occur if the photon energy equals or exceeds 0.6 eV, the smallest amount of energy needed to dissociate an AgBr molecule in the film. (a) What is the greatest wavelength of light that can be recorded by the film? (b) In what region of the electromagnetic spectrum is this wavelength located?

Answer:

(a) 2.1 μm; (b) infrared

•2 How fast must an electron move to have a kinetic energy equal to the photon energy of sodium light at wavelength 590 nm?

•3 At what rate does the Sun emit photons? For simplicity, assume that the Sun's entire emission at the rate of 3.9×10^{26} W is at the single wavelength of 550 nm.

Answer:

1.0×10^{45} photons/s

•4 A helium–neon laser emits red light at wavelength λ = 633 nm in a beam of diameter 3.5 mm and at an energy-emission rate of 5.0 mW A detector in the beam's path totally absorbs the beam. At what rate per unit area does the detector absorb photons?

•5 The meter was once defined as 1 650 763.73 wavelengths of the orange light emitted by a source containing krypton-86 atoms. What is the photon energy of that light?

Answer:

2.047 eV

•6 The yellow-colored light from a highway sodium lamp is brightest at a wavelength of 589 nm. What is the photon energy for light at that wavelength?

••7 A light detector (your eye) has an area of 2.00×10^{-6} m^2 and absorbs 80% of the incident light, which is at wavelength 500 nm. The detector faces an isotropic source, 3.00 m from the source. If the detector absorbs photons at the rate of exactly 4.000 s^{-1}, at what power does the emitter emit light?

Answer:

1.1×10^{-10}W

••8 The beam emerging from a 1.5 W argon laser (λ = 515 nm) has a diameter *d* of 3.0 mm. The beam is focused by a lens system with an effective focal length f_L of 2.5 mm. The focused beam strikes a totally absorbing screen, where it forms a circular diffraction pattern whose central disk has a radius *R* given by $1.22 f_L \lambda / d$. It can be shown that 84% of the incident energy ends up within this central disk. At what rate are photons absorbed by the screen in the central disk of the diffraction pattern?

••9 GO A 100 W sodium lamp (λ = 589 nm) radiates energy uniformly in all directions. (a) At what rate are photons emitted by the lamp? (b) At what distance from the lamp will a totally absorbing screen absorb photons at the rate of 1.00 photon/$cm^2 \cdot s$? (c) What is the photon flux (photons per unit area per unit time) on a small screen 2.00 m from the lamp?

Answer:

(a) 2.96×10^{20}photons/s; (b) 4.86×10^{7}m; (c) 5.89×10^{18} photons/$m^2 \cdot s$

••10A satellite in Earth orbit maintains a panel of solar cells of area 2.60 m^2 perpendicular to the direction of the Sun's light rays. The intensity of the light at the panel is 1.39 kW/m^2. (a) At what rate does solar energy arrive at the panel? (b) At what rate are solar photons absorbed by the panel? Assume that the solar radiation is monochromatic, with a wavelength of 550 nm, and that all the solar radiation striking the panel is absorbed. (c) How long would it take for a “mole of photons” to be absorbed by the panel?

••11 SSM WWW An ultraviolet lamp emits light of wavelength 400 nm at the rate of 400 W An infrared lamp emits light of wavelength 700 nm, also at the rate of 400 W. (a) Which lamp emits photons at the greater rate and (b) what is that greater rate?

Answer:

(a) infrared; (b) 1.4×10^{21} photons/s

••12Under ideal conditions, a visual sensation can occur in the human visual system if light of wavelength 550 nm is absorbed by the eye's retina at a rate as low as 100 photons per second. What is the corresponding rate at which energy is absorbed by the retina?

••13A special kind of lightbulb emits monochromatic light of wavelength 630 nm. Electrical energy is supplied to it at the rate of 60 W, and the bulb is 93% efficient at converting that energy to light energy. How many photons are emitted by the bulb during its lifetime of 730 h?

Answer:

4.7×10^{26} photons

••14 GO A light detector has an absorbing area of 2.00×10^{-6} m^2 and absorbs 50% of the incident light, which is at wavelength 600 nm. The detector faces an isotropic source, 12.0 m from the source. The energy E emitted by the source versus time t is given in Fig. 38-24 (E_s = 7.2 nJ, t_s = 2.0 s). At what rate are photons absorbed by the detector?

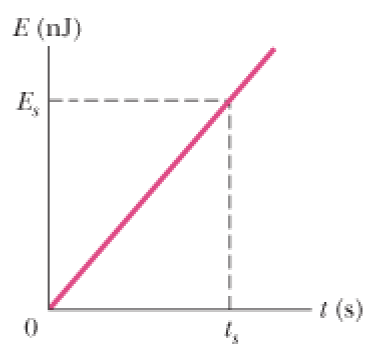


Figure 38-24 Problem 14.

sec. 38-3 The Photoelectric Effect

•15 SSM Light strikes a sodium surface, causing photoelectric emission. The stopping potential for the ejected electrons is 5.0 V, and the work function of sodium is 2.2 eV. What is the wavelength of the incident light?

Answer:

170 nm

•16 Find the maximum kinetic energy of electrons ejected from a certain material if the material's work function is 2.3 eV and the frequency of the incident radiation is 3.0×10^{15} Hz.

•17 The work function of tungsten is 4.50 eV. Calculate the speed of the fastest electrons ejected from a tungsten surface when light whose photon energy is 5.80 eV shines on the surface.

Answer:

676 km/s

•18 You wish to pick an element for a photocell that will operate via the photoelectric effect with visible light. Which of the following are suitable (work functions are in parentheses): tantalum (4.2 eV), tungsten (4.5 eV), aluminum (4.2 eV), barium (2.5 eV), lithium (2.3 eV)?

••19 (a) If the work function for a certain metal is 1.8 eV, what is the stopping potential for electrons ejected from the metal when light of wavelength 400 nm shines on the metal? (b) What is the maximum speed of the ejected electrons?

Answer:

1.3 V; (b) 6.8×10^2km/s

••20 Suppose the *fractional efficiency* of a cesium surface (with work function 1.80 eV) is 1.0×10^{-16}; that is, on average one electron is ejected for every 10^{16} photons that reach the surface. What would be the current of electrons ejected from such a surface if it were illuminated with 600 nm light from a 2.00 mW laser and all the ejected electrons took part in the charge flow?

••21 GO X rays with a wavelength of 71 pm are directed onto a gold foil and eject tightly bound electrons from the gold atoms. The ejected electrons then move in circular paths of radius r in a region of uniform magnetic field $\vec{B}$. For the fastest of the ejected electrons, the product Br is equal to 1.88×10^{-4} T · m. Find (a) the maximum kinetic energy of those electrons and (b) the work done in removing them from the gold atoms.

Answer:

(a) 3.1 keV; (b) 14 keV

••22 The wavelength associated with the cutoff frequency for silver is 325 nm. Find the maximum kinetic energy of electrons ejected from a silver surface by ultraviolet light of wavelength 254 nm.

••23 SSM Light of wavelength 200 nm shines on an aluminum surface; 4.20 eV is required to eject an electron. What is the kinetic energy of (a) the fastest and (b) the slowest ejected electrons? (c) What is the stopping potential for this situation? (d) What is the cutoff wavelength for aluminum?

Answer:

(a) 2.00 eV; (b) 0; (c) 2.00 V; (d) 295 nm

••24 In a photoelectric experiment using a sodium surface, you find a stopping potential of 1.85 V for a wavelength of 300 nm and a stopping potential of 0.820 V for a wavelength of 400 nm. From these data find (a) a value for the Planck constant, (b) the work function Φ for sodium, and (c) the cutoff wavelength λ_0 for sodium.

••25 The stopping potential for electrons emitted from a surface illuminated by light of wavelength 491 nm is 0.710 V. When the incident wavelength is changed to a new value, the stopping potential is 1.43 V. (a) What is this new wavelength? (b) What is the work function for the surface?

Answer:

(a) 382 nm; (b) 1.82 eV

••26 An orbiting satellite can become charged by the photoelectric effect when sunlight ejects electrons from its outer surface. Satellites must be designed to minimize such charging because it can ruin the sensitive microelectronics. Suppose a satellite is coated with platinum, a metal with a very large work function (Φ = 5.32 eV). Find the longest wavelength of incident sunlight that can eject an electron from the platinum.

sec. 38-4 Photons Have Momentum

•27 SSM Light of wavelength 2.40 pm is directed onto a target containing free electrons. (a) Find the wavelength of light scattered at 30.0° from the incident direction. (b) Do the same for a scattering angle of 120°.

Answer:

(a) 2.73 pm; (b) 6.05 pm

•28 (a) In MeV/*c*, what is the magnitude of the momentum associated with a photon having an energy equal to the electron rest energy? What are the (b) wavelength and (c) frequency of the corresponding radiation?

•29 What (a) frequency, (b) photon energy, and (c) photon momentum magnitude (in keV/*c*) are associated with x rays having wavelength 35.0 pm?

Answer:

(a) 8.57×10^{18} Hz; (b) 3.55×10^{4} eV; (c) 35.4 keV/c

••30 What is the maximum wavelength shift for a Compton collision between a photon and a free *proton*?

••31 What percentage increase in wavelength leads to a 75% loss of photon energy in a photon-free electron collision?

Answer:

300%

••32 X rays of wavelength 0.0100 nm are directed in the positive direction of an *x* axis onto a target containing loosely bound electrons. For Compton scattering from one of those electrons, at an angle of 180°, what are (a) the Compton shift, (b) the corresponding change in photon energy, (c) the kinetic energy of the recoiling electron, and (d) the angle between the positive direction of the *x* axis and the electron's direction of motion?

••33 Calculate the percentage change in photon energy during a collision like that in Fig. 38-5 for = 90° and for radiation in (a) the microwave range, with λ = 3.0 cm; (b) the visible range, with λ = 500 nm; (c) the x-ray range, with λ = 25 pm; and (d) the gamma-ray range, with a gamma photon energy of 1.0 MeV. (e) What are your conclusions about the feasibility of detecting the Compton shift in these various regions of the electromagnetic spectrum, judging solely by the criterion of energy loss in a single photon-electron encounter?

Answer:

(a) $-8.1 \times 10^{-9}\%$; (b) $-4.9 \times 10^{-4}\%$; (c) - 8.9%; (d) -66%

••34 A photon undergoes Compton scattering off a stationary free electron. The photon scatters at 90.0° from its initial direction; its initial wavelength is 3.00×10^{-12} m. What is the electron's kinetic energy?

••35 Calculate the Compton wavelength for (a) an electron and (b) a proton. What is the photon energy for an electromagnetic wave with a wavelength equal to the Compton wavelength of (c) the electron and (d) the proton?

Answer:

(a) 2.43 pm;(b) 1.32 fm;(c) 0.511 MeV; (d) 939 MeV

••36 Gamma rays of photon energy 0.511 MeV are directed onto an aluminum target and are scattered in various directions by loosely bound electrons there. (a) What is the wavelength of the incident gamma rays? (b) What is the wavelength of gamma rays scattered at 90.0° to the incident beam? (c) What is the photon energy of the rays scattered in this direction?

••37 Consider a collision between an x-ray photon of initial energy 50.0 keV and an electron at rest, in which the photon is scattered backward and the electron is knocked forward. (a) What is the energy of the back-scattered photon? (b) What is the kinetic energy of the electron?

Answer:

(a) 41.8 keV;(b) 8.2 keV

••38 Show that when a photon of energy E is scattered from a free electron at rest, the maximum kinetic energy of the recoiling electron is given by

$$K_{\max} = \frac{E^2}{E + mc^2 / 2}.$$

••39 Through what angle must a 200 keV photon be scattered by a free electron so that the photon loses 10% of its energy?

Answer:

44°

••40 What is the maximum kinetic energy of electrons knocked out of a thin copper foil by Compton scattering of an incident beam of 17.5 keV x rays? Assume the work function is negligible.

••41 What are (a) the Compton shift $\Delta\lambda$, (b) the fractional Compton shift $\Delta\lambda/\lambda$, and (c) the change ΔE in photon energy for light of wavelength λ = 590 nm scattering from a free, initially stationary electron if the scattering is at 90° to the direction of the incident beam? What are (d) $\Delta\lambda$, (e) $\Delta\lambda/\lambda$, and (f) ΔE for 90° scattering for photon energy 50.0 keV (x-ray range)?

Answer:

(a) 2.43 pm; (b) 4.11×10^{-6}; (c) -8.67×10^{-6} eV; (d) 2.43 pm; (e) 9.78×10^{-2}; (f) -4.45 keV

sec. 38-6 Electrons and Matter Waves

•42 Calculate the de Broglie wavelength of (a) a 1.00 keV electron, (b) a 1.00 keV photon, and (c) a 1.00 keV neutron.

•43 SSM In an old-fashioned television set, electrons are accelerated through a potential difference of 25.0 kV. What is the de Broglie wavelength of such electrons? (Relativity is not needed.)

Answer:

7.75 pm

••44 The smallest dimension (*resolving power*) that can be resolved by an electron microscope is equal to the de Broglie wavelength of its electrons. What accelerating voltage would be required for the electrons to have the same resolving power as could be obtained using 100 keV gamma rays?

••45 SSM WWW Singly charged sodium ions are accelerated through a potential difference of 300 V. (a) What is the momentum acquired by such an ion? (b) What is its de Broglie wavelength?

Answer:

(a) 1.9×10^{-21} kg · m/s;(b) 346 fm

••46 Electrons accelerated to an energy of 50 GeV have a de Broglie wavelength λ small enough for them to probe the structure within a target nucleus by scattering from the structure. Assume that the energy is so large that the extreme relativistic relation $p = E/c$ between momentum magnitude p and energy E applies. (In this extreme situation, the kinetic energy of an electron is much greater than its rest energy.) (a) What is λ? (b) If the target nucleus has radius $R = 5.0$ fm, what is the ratio R/λ?

••47 SSM The wavelength of the yellow spectral emission line of sodium is 590 nm. At what kinetic energy would an electron have that wavelength as its de Broglie wavelength?

Answer:

4.3 μeV

••48 A stream of protons, each with a speed of $0.9900c$, are directed into a two-slit experiment where the slit separation is 4.00×10^{-9} m. A two-slit interference pattern is built up on the viewing screen. What is the angle between the center of the pattern and the second minimum (to either side of the center)?

••49 What is the wavelength of (a) a photon with energy 1.00 eV, (b) an electron with energy 1.00 eV, (c) a photon of energy 1.00 GeV, and (d) an electron with energy 1.00 GeV?

Answer:

(a) 1.24 μm; (b) 1.22 nm; (c) 1.24 fm; (d) 1.24 fm

••50 An electron and a photon each have a wavelength of 0.20 nm. What is the momentum (in kg · m/s) of the (a) electron and (b) photon? What is the energy (in eV) of the (c) electron and (d) photon?

••51 The highest achievable resolving power of a microscope is limited only by the wavelength used; that is, the smallest item that can be distinguished has dimensions about equal to the wavelength. Suppose one wishes to “see” inside an atom. Assuming the atom to have a diameter of 100 pm, this means that one must be able to resolve a width of, say, 10 pm. (a) If an electron microscope is used, what minimum electron energy is required? (b) If a light microscope is used, what minimum photon energy is required? (c) Which microscope seems more practical? Why?

Answer:

(a) 15 keV; (b) 120 keV

••52 The existence of the atomic nucleus was discovered in 1911 by Ernest Rutherford, who properly interpreted some experiments in which a beam of alpha particles was scattered from a metal foil of atoms such as gold. (a) If the alpha particles had a kinetic energy of 7.5 MeV, what was their de Broglie wavelength? (b) Explain whether the wave nature of the incident alpha particles should have been taken into account in interpreting these experiments. The mass of an alpha particle is 4.00 u (atomic mass units), and its distance of closest approach to the nuclear center in these experiments was about 30 fm. (The wave nature of matter was not postulated until more than a decade after these crucial experiments were first performed.)

••53 A nonrelativistic particle is moving three times as fast as an electron. The ratio of the de Broglie wavelength of the particle to that of the electron is 1.813×10^{-4}. By calculating its mass, identify the particle.

Answer:

neutron

••54 What are (a) the energy of a photon corresponding to wavelength 1.00 nm, (b) the kinetic energy of an electron with de Broglie wavelength 1.00 nm, (c) the energy of a photon corresponding to wavelength 1.00 fm, and (d) the kinetic energy of an electron with de Broglie wavelength 1.00 fm?

•••55 GO If the de Broglie wavelength of a proton is 100 fm, (a) what is the speed of the proton and (b) through what electric potential would the proton have to be accelerated to acquire this speed?

Answer:

(a) 3.96×10^6 m/s; (b) 81.7 kV

sec. 38-7 Schrödinger's Equation

••56 Suppose we put $A = 0$ in Eq. 38-17 and relabeled B as ψ_0. (a) What would the resulting wave function then describe? (b) How, if at all, would Fig. 38-12 be altered?

••57 SSM The function $\psi(x)$ displayed in Eq. 38-19 describes a free particle, for which we assumed that $U(x) = 0$ in Schrödinger's equation (Eq. 38-15). Assume now that $U(x) = U_0$ = a constant in that equation. Show that Eq. 38-19 is still a solution of Schrödinger's equation, with

$$k = \frac{2\pi}{h}\sqrt{2m(E - U_0)}$$

now giving the angular wave number k of the particle.

••58 In Eq. 38-18 keep both terms, putting $A = B = \psi_0$. The equation then describes the superposition of two matter waves of equal amplitude, traveling in opposite directions. (Recall that this is the condition for a standing wave.) (a) Show that $|\Psi(x, t)|^2$ is then given by

$$\left|\Psi(x, t)\right|^2 = 2\psi_0^2[1 + \cos 2kx].$$

(b) Plot this function, and demonstrate that it describes the square of the amplitude of a standing matter wave. (c) Show that the nodes of this standing wave are located at

$$x = (2n+1)\left(\frac{1}{4}\lambda\right), \quad \text{where } n = 0, 1, 2, 3, \ldots$$

and λ is the de Broglie wavelength of the particle. (d) Write a similar expression for the most probable locations of the particle.

••59 Show that Eq. 38-17 is indeed a solution of Eq. 38-16 by substituting $\psi(x)$ and its second derivative into Eq. 38-16 and noting that an identity results.

••60 (a) Write the wave function $\psi(x)$ displayed in Eq. 38-19 in the form $\psi(x) = a + ib$, where a and b are real quantities. (Assume that ψ_0 is real.) (b) Write the time-dependent wave function $\Psi(x, t)$ that corresponds to $\psi(x)$ written in this form.

••61 SSM Show that the angular wave number k for a nonrelativistic free particle of mass m can be written as

$$k = \frac{2\pi\sqrt{2mK}}{h},$$

in which K is the particle's kinetic energy.

••62 (a) Let $n = a + ib$ be a complex number, where a and b are real (positive or negative) numbers. Show that the product nn^* is always a positive real number. (b) Let $m = c + id$ be another complex number. Show that $|nm| = |n|\,|m|$.

sec. 38-8 Heisenberg's Uncertainty Principle

•63 The uncertainty in the position of an electron along an x axis is given as 50 pm, which is about equal to the radius of a hydrogen atom. What is the least uncertainty in any simultaneous measurement of the momentum component p_x of this electron?

Answer:

2.1×10^{-24} kg · m/s

••64 You will find in Chapter 39 that electrons cannot move in definite orbits within atoms, like the planets in our solar system. To see why, let us try to "observe" such an orbiting electron by using a light microscope to measure the electron's presumed orbital position with a precision of, say, 10 pm (a typical atom has a radius of about 100 pm). The wavelength of the light used in the microscope must then be about 10 pm. (a) What would be the photon energy of this light? (b) How much energy would such a photon impart to an electron in a head-on collision? (c) What do these results tell you about the possibility of "viewing" an atomic electron at two or more points along its presumed orbital path? (*Hint:* The outer electrons of atoms are bound to the atom by energies of only a few electron-volts.)

••65 Figure 38-12 shows a case in which the momentum component p_x of a particle is fixed so that $\Delta p_x = 0$; then, from Heisenberg's uncertainty principle (Eq. 38-20), the position x of the particle is completely unknown. From the same principle it follows that the opposite is also true; that is, if the position of a particle is exactly known ($\Delta x = 0$), the uncertainty in its momentum is infinite.

Consider an intermediate case, in which the position of a particle is measured, not to infinite precision, but to within a distance of $\lambda/2\pi$, where A is the particle's de Broglie wavelength. Show that the uncertainty in the (simultaneously measured) momentum component is then equal to the component itself; that is, $\Delta p_x = p$. Under these circumstances, would a measured momentum of zero surprise you? What about a measured momentum of 0.5*p*? Of 2*p*? Of 12*p*?

sec. 38-9 Barrier Tunneling

••66 Consider a potential energy barrier like that of Fig. 38-15 but whose height U_b is 6.0 eV and whose thickness L is 0.70 nm. What is the energy of an incident electron whose transmission coefficient is 0.0010?

••67 A 3.0 MeV proton is incident on a potential energy barrier of thickness 10 fm and height 10 MeV. What are (a) the transmission coefficient T, (b) the kinetic energy K, the proton will have on the other side of the barrier if it tunnels through the barrier, and (c) the kinetic energy K_r it will have if it reflects from the barrier? A 3.0 MeV deuteron (the same charge but twice the mass as a proton) is incident on the same barrier. What are (d) T, (e) K, and (f) K_r?

Answer:

(a) 9.02×10^{-6};(b) 3.0 MeV; (c) 3.0 MeV; (d) 7.33×10^{-8}; (e) 3.0 MeV; (f) 3.0 MeV

••68 GO (a) Suppose a beam of 5.0 eV protons strikes a potential energy barrier of height 6.0 eV and thickness 0.70 nm, at a rate equivalent to a current of 1000 A. How long would you have to wait—on average—for one proton to be transmitted? (b) How long would you have to wait if the beam consisted of electrons rather than protons?

••69 SSM WWW An electron with total energy $E = 5.1$ eV approaches a barrier of height $U_b = 6.8$ eV and thickness $L = 750$ pm. What percentage change in the transmission coefficient T occurs for a 1.0% change in (a) the barrier height, (b) the barrier thickness, and (c) the kinetic energy of the incident electron?

Answer:

(a) -20%; (b) -10%; (c) +15%

Additional Problems

70 Figure 38-12 shows that because of Heisenberg's uncertainty principle, it is not possible to assign an x coordinate to the position of a free electron moving along an x axis. (a) Can you assign a y or a z coordinate? (*Hint:* The momentum of the electron has no y or z component.) (b) Describe the extent of the matter wave in three dimensions.

71 A spectral emission line is electromagnetic radiation that is emitted in a wavelength range narrow enough to be taken as a single wavelength. One such emission line that is important in astronomy has a wavelength of 21 cm. What is the photon energy in the electromagnetic wave at that wavelength?

Answer:

5.9 μeV

72 Using the classical equations for momentum and kinetic energy, show that an electron's de Broglie wavelength in nanometers can be written as $\lambda = 1.226 / \sqrt{K}$, in which K is the electron's kinetic energy in electron-volts.

73 Derive Eq. 38-11, the equation for the Compton shift, from Eqs. 38-8, 38-9, and 38-10 by eliminating v and θ.

74 Neutrons in thermal equilibrium with matter have an average kinetic energy of $(3/2)kT$, where k is the Boltzmann constant and T, which may be taken to be 300 K, is the temperature of the environment of the neutrons. (a) What is the average kinetic energy of such a neutron? (b) What is

the corresponding de Broglie wavelength?

75 Consider a balloon filled with helium gas at room temperature and atmospheric pressure. Calculate (a) the average de Broglie wavelength of the helium atoms and (b) the average distance between atoms under these conditions. The average kinetic energy of an atom is equal to (3/2)kT, where k is the Boltzmann constant. (c) Can the atoms be treated as particles under these conditions? Explain.

Answer:

(a) 73 pm; (b) 3.4 nm; (c) yes, their average de Broglie wavelength is smaller than their average separation

76 In about 1916, R. A. Millikan found the following stopping-potential data for lithium in his photoelectric experiments:

Wavelength (nm)	433.9	404.7	365.0	312.5	253.5
Stopping potential (V)	0.55	0.73	1.09	1.67	2.57

Use these data to make a plot like Fig. 38-2 (which is for sodium) and then use the plot to find (a) the Planck constant and (b) the work function for lithium.

77 Show that $|\psi|^2 = |\Psi|^2$, with ψ and Ψ related as in Eq. 38-14. That is, show that the probability density does not depend on the time variable.

78 Show that $\Delta E/E$, the fractional loss of energy of a photon during a collision with a particle of mass m, is given by

$$\frac{\Delta E}{E} = \frac{hf'}{mc^2}(1 - \cos\phi),$$

where E is the energy of the incident photon, f' is the frequency of the scattered photon, and is defined as in Fig. 38-5.

79 A bullet of mass 40 g travels at 1000 m/s. Although the bullet is clearly too large to be treated as a matter wave, determine what Eq. 38-13 predicts for the de Broglie wavelength of the bullet at that speed.

Answer:

1.7×10^{-35} m

80 (a) The smallest amount of energy needed to eject an electron from metallic sodium is 2.28 eV. Does sodium show a photoelectric effect for red light, with λ = 680 nm? (That is, does the light cause electron emission?) (b) What is the cutoff wavelength for photoelectric emission from sodium? (c) To what color does that wavelength correspond?

81 SSM Imagine playing baseball in a universe (not ours!) where the Planck constant is 0.60 J · s and thus quantum physics affects macroscopic objects. What would be the uncertainty in the position of a 0.50 kg baseball that is moving at 20 m/s along an axis if the uncertainty in the speed is 1.0 m/s?

Answer:

0.19 m

82 An electron of mass m and speed v "collides" with a gamma-ray photon of initial energy hf_0, as measured in the laboratory frame. The photon is scattered in the electron's direction of travel.

Verify that the energy of the scattered photon, as measured in the laboratory frame, is

$$E = hf_0\left(1 + \frac{2hf_0}{mc^2}\sqrt{\frac{1+v/c}{1-v/c}}\right)^{-1}.$$

83 Show, by analyzing a collision between a photon and a free electron (using relativistic mechanics), that it is impossible for a photon to transfer all its energy to a free electron (and thus for the photon to vanish).

84 A 1500 kg car moving at 20 m/s approaches a hill that is 24 m high and 30 m long. Although the car and hill are clearly too large to be treated as matter waves, determine what Eq. 38-21 predicts for the transmission coefficient of the car, as if it could tunnel through the hill as a matter wave. Treat the hill as a potential energy barrier where the potential energy is gravitational.

CHAPTER 39 MORE ABOUT MATTER WAVES

39-1 What is Physics?

One of the long-standing goals of physics has been to understand the nature of atoms. Early in the 20th century nobody knew how the electrons in an atom are arranged, what their motions are, how atoms emit or absorb light, or even why atoms are stable. Without this knowledge it was not possible to understand how atoms combine to form molecules or stack up to form solids. As a consequence, the foundations of chemistry—including biochemistry, which underlies the nature of life itself—were more or less a mystery.

In 1926, all these questions and many others were answered with the development of quantum physics. Its basic premise is that moving electrons, protons, and particles of any kind are best viewed as matter waves, whose motions are governed by Schrödinger's equation. Although quantum theory also applies to massive particles, there is no point in treating baseballs, automobiles, planets, and such objects with quantum theory. For such massive, slow-moving objects, Newtonian physics and quantum physics yield the same answers.

Before we can apply quantum physics to the problem of atomic structure, we need to develop some insights by applying quantum ideas in a few simpler situations. Some of these situations may seem simplistic and unreal, but they allow us to discuss the basic principles of the quantum physics of atoms without having to deal with the often overwhelming complexity of atoms. Besides, with the advances in nanotechnology, situations that were previously found only in textbooks are now being produced in laboratories and put to use in modern electronics and materials science applications. We are on the threshold of being able to use nanometer-scale constructions called *quantum corrals* and *quantum dots* to create “designer atoms” whose properties can be manipulated in the laboratory. For both natural atoms and these artificial ones, the starting point in our discussion is the wave nature of an electron.

39-2 String Waves and Matter Waves

In Chapter 16 we saw that waves of two kinds can be set up on a stretched string. If the string is so long that we can take it to be infinitely long, we can set up a *traveling wave* of essentially any frequency. However, if the stretched string has only a finite length, perhaps because it is rigidly clamped at both ends, we can set up only *standing waves* on it; further, these standing waves can have only discrete frequencies. In other words, confining the wave to a finite region of space leads to *quantization* of the motion—to the existence of discrete *states* for the wave, each state with a sharply defined frequency.

This observation applies to waves of all kinds, including matter waves. For matter waves, however, it is more convenient to deal with the energy E of the associated particle than with the frequency f of the wave. In all that follows we shall focus on the matter wave associated with an electron, but the results apply to any confined matter wave.

Consider the matter wave associated with an electron moving in the positive x direction and subject to no net force—a so-called *free particle*. The energy of such an electron can have any reasonable value, just as a wave traveling along a stretched string of infinite length can have any reasonable frequency.

Consider next the matter wave associated with an atomic electron, perhaps the *valence* (least tightly bound) electron. The electron—held within the atom by the attractive Coulomb force between it and the positively charged nucleus—is not a free particle. It can exist only in a set of discrete states, each having a discrete energy E. This sounds much like the discrete states and quantized frequencies that are available to a stretched string of finite length. For matter waves, then, as for all other kinds of waves, we may state a **confinement principle**:

> Confinement of a wave leads to quantization—that is, to the existence of discrete states with discrete energies. The wave can have only those energies.

39-3 Energies of a Trapped Electron

One-Dimensional Traps

Here we examine the matter wave associated with a nonrelativistic electron confined to a limited region of space. We do so by analogy with standing waves on a string of finite length, stretched along an x axis and confined between rigid supports. Because the supports are rigid, the two ends of the string are nodes, or points at which the string is always at rest. There may be other nodes along the string, but these two must always be present, as Fig. 16-20 shows.

The states, or discrete standing wave patterns in which the string can oscillate, are those for which the length L of the string is equal to an integer number of half-wavelengths. That is, the string can occupy only states for which

$$L = \frac{n\lambda}{2}, \quad \text{for } n = 1, 2, 3, \ldots . \qquad (39\text{-}1)$$

Each value of n identifies a state of the oscillating string; using the language of quantum physics, we can call the integer n a **quantum number**.

For each state of the string permitted by Eq. 39-1, the transverse displacement of the string at any position x along the string is given by

$$y_n(x) = A \sin\left(\frac{n\pi}{L}x\right), \quad \text{for } n = 1, 2, 3, \ldots, \qquad (39\text{-}2)$$

in which the quantum number n identifies the oscillation pattern and A depends on the time at which you inspect the string. (Equation 39-2 is a short version of Eq. 16-60.) We see that for all values of n and for all times, there is a point of zero displacement (a node) at $x = 0$ and at $x = L$, as there must be. Figure 39-19 shows time exposures of such a stretched string for $n = 2{,}3$, and 4.

Now let us turn our attention to matter waves. Our first problem is to physically confine an electron that is moving along the x axis so that it remains within a finite segment of that axis. Figure 39-1 shows a conceivable one-dimensional *electron trap*. It consists of two semi-infinitely long cylinders, each of which has an electric potential approaching $-\infty$; between them is a hollow cylinder of length L, which has an electric potential of zero. We put a single electron into this central cylinder to trap it.

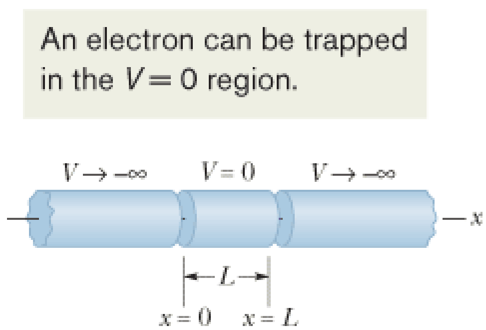


Figure 39-1 The elements of an idealized "trap" designed to confine an electron to the central cylinder. We take the semi-infinitely long end cylinders to be at an infinitely great negative potential and the central cylinder to be at zero potential.

The trap of Fig. 39-1 is easy to analyze but is not very practical. Single electrons *can*, however, be trapped in the laboratory with traps that are more complex in design but similar in concept. At the University of Washington, for example, a single electron has been held in a trap for months on end, permitting scientists to make extremely precise measurements of its properties.

Finding the Quantized Energies

Figure 39-2 shows the potential energy of the electron as a function of its position along the x axis of the idealized trap of Fig. 39-1. When the electron is in the central cylinder, its potential energy U (= -

eV) is zero because there the potential *V* is zero. If the electron could get outside this region, its potential energy would be positive and of infinite magnitude because there $V \rightarrow -\infty$. We call the potential energy pattern of Fig. 39-2 an **infinitely deep potential energy well** or, for short, an *infinite potential well*. It is a "well" because an electron placed in the central cylinder of Fig. 39-1 cannot escape from it. As the electron approaches either end of the cylinder, a force of essentially infinite magnitude reverses the electron's motion, thus trapping it. Because the electron can move along only a single axis, this trap can be called a *one-dimensional infinite potential well*.

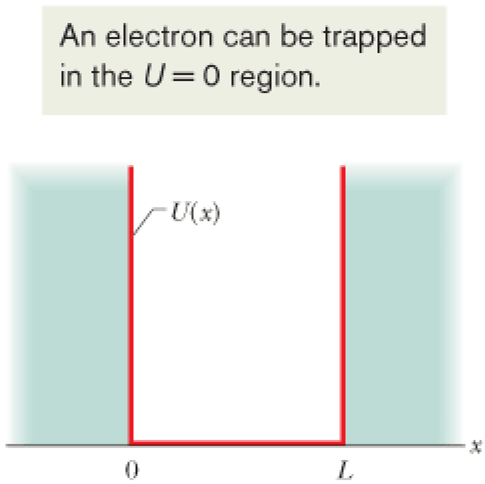


Figure 39-2 The electric potential energy $U(x)$ of an electron confined to the central cylinder of the idealized trap of Fig. 39-1. We see that $U = 0$ for $0 < x < L$, and $U \rightarrow \infty$ for $x < 0$ and $x > L$.

Just like the standing wave in a length of stretched string, the matter wave describing the confined electron must have nodes at $x = 0$ and $x = L$. Moreover, Eq. 39-1 applies to such a matter wave if we interpret λ in that equation as the de Broglie wavelength associated with the moving electron.

The de Broglie wavelength λ is defined in Eq. 38-13 as $\lambda = h/p$, where p is the magnitude of the electron's momentum. Because the electron is nonrelativistic, this momentum magnitude p is related to the kinetic energy K by $p = \sqrt{2mK}$, where m is the mass of the electron. For an electron moving within the central cylinder of Fig. 39-1, where $U = 0$, the total (mechanical) energy E is equal to the kinetic energy. Hence, we can write the de Broglie wavelength of this electron as

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}. \qquad (39\text{-}3)$$

If we substitute Eq. 39-3 into Eq. 39-1 and solve for the energy E, we find that E depends on n according to

$$E_n = \left(\frac{h^2}{8mL^2}\right)n^2, \qquad \text{for } n = 1, 2, 3, \ldots. \qquad (39\text{-}4)$$

The positive integer n here is the quantum number of the electron's quantum state in the trap.

Equation 39-4 tells us something important: Because the electron is confined to the trap, it can have only the energies given by the equation. It *cannot* have an energy that is, say, halfway between the values for $n = 1$ and $n = 2$. Why this restriction? Because an electron is a matter wave. Were it, instead, a particle as assumed in classical physics, it could have *any* value of energy while it is confined to the trap.

Figure 39-3 is a graph showing the lowest five allowed energy values for an electron in an infinite well with L = 100 pm (about the size of a typical atom). The values are called *energy levels*, and they are drawn in Fig. 39-3 as levels, or steps, on a ladder, in an *energy-level diagram*. Energy is plotted vertically; nothing is plotted horizontally.

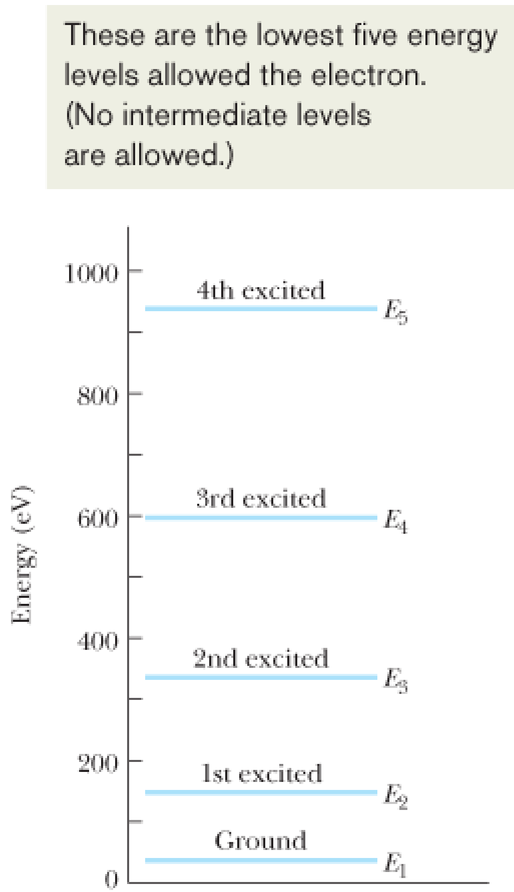


Figure 39-3 Several of the allowed energies given by Eq. 39-4 for an electron confined to the infinite well of Fig. 39-2. Here width L = 100 pm. Such a plot is called an *energy-level diagram*.

The quantum state with the lowest possible energy level E_1 allowed by Eq. 39-4, with quantum number n = 1, is called the *ground state* of the electron. The electron tends to be in this lowest energy state. All the quantum states with greater energies (corresponding to quantum numbers $n = 2$ or greater) are called *excited states* of the electron. The state with energy level E_2, for quantum number n = 2, is called the *first excited state* because it is the first of the excited states as we move up the energy-level diagram. The other states have similar names.

Energy Changes

A trapped electron tends to have the lowest allowed energy and thus to be in its ground state. It can be changed to an excited state (in which it has greater energy) only if an external source provides the additional energy that is required for the change. Let E_{low} be the initial energy of the electron and E_{high}

be the greater energy in a state that is higher on its energy-level diagram. Then the amount of energy that is required for the electron's change of state is

$$\Delta E = E_{\text{high}} - E_{\text{low}}. \tag{39-5}$$

An electron that receives such energy is said to make a *quantum jump* (or *transition*), or to be *excited* from the lower-energy state to the higher-energy state. Figure 39-4*a* represents a quantum jump from the ground state (with energy level E_1) to the third excited state (with energy level E_4). As shown, the jump *must* be from one energy level to another, but it can bypass one or more intermediate energy levels.

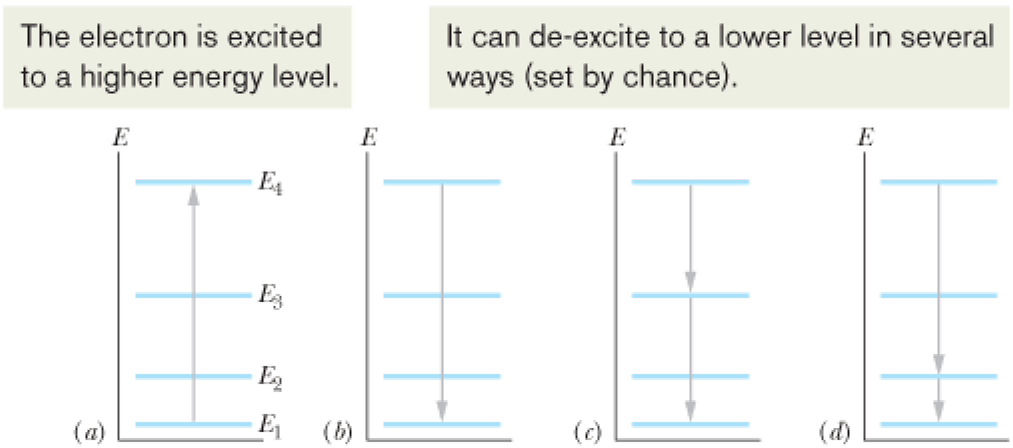


Figure 39-4(*a*) Excitation of a trapped electron from the energy level of its ground state to the level of its third excited state. (*b*)–(*d*) Three of four possible ways the electron can de-excite to return to the energy level of its ground state. (Which way is not shown?)

One way an electron can gain energy to make a quantum jump up to a greater energy level is to absorb a photon. However, this absorption and quantum jump can occur only if the following condition is met:

If a confined electron is to absorb a photon, the energy hf of the photon must equal the energy difference ΔE between the initial energy level of the electron and a higher level.

Thus, excitation by the absorption of light requires that

$$hf = \Delta E = E_{\text{high}} - E_{\text{low}}. \tag{39-6}$$

When an electron reaches an excited state, it does not stay there but quickly *de-excites* by decreasing its energy. Figures 39-4*b* to *d* represent some of the possible quantum jumps down from the energy level of the third excited state. The electron can reach its ground-state level either with one direct quantum jump (Fig. 39-4*b*) or with shorter jumps via intermediate levels (Figs. 39-4*c* and 39-4*d*).

An electron can decrease its energy by emitting a photon but only this way:

If a confined electron emits a photon, the energy hf of that photon must equal the energy difference ΔE between the initial energy level of the electron and a lower level.

CHECKPOINT 1

Rank the following pairs of quantum states for an electron confined to an infinite well according to the energy differences between the states, greatest first: (a) $n = 3$ and $n = 1$, (b) $n = 5$ and $n = 4$, (c) $n = 4$ and $n = 3$.

Answer:

b, a, c

Thus, Eq. 39-6 applies to both the absorption and the emission of light by a confined electron. That is, the absorbed or emitted light can have only certain values of hf and thus only certain values of frequency f and wavelength λ.

Aside: Although Eq. 39-6 and what we have discussed about photon absorption and emission can be applied to physical (real) electron traps, they actually cannot be applied to one-dimensional (unreal) electron traps. The reason involves the need to conserve angular momentum in a photon absorption or emission process. In this book, we shall neglect that need and use Eq. 39-6 even for one-dimensional traps.

Energy levels in a 1D infinite potential well

(a) An electron is confined to a one-dimensional, infinitely deep potential energy well of width $L = 100$ pm.

What is the smallest amount of energy the electron can have?

KEY IDEA

Confinement of the electron (a matter wave) to the well leads to quantization of its energy. Because the well is infinitely deep, the allowed energies are given by Eq. 39-4 ($E_n = (h^2/8mL^2)n^2$), with the quantum number n a positive integer.

Lowest energy level:

Here, the collection of constants in front of n^2 in Eq. 39-4 is evaluated as

$$\frac{h^2}{8mL^2} = \frac{\left(6.63 \times 10^{-34} \mathrm{J \cdot s}\right)^2}{\left(8\right)\left(9.11 \times 10^{-31} \mathrm{kg}\right)\left(100 \times 10^{-12} \mathrm{m}\right)^2} \tag{39-7}$$

$$= 6.031 \times 10^{-18} \mathrm{J}.$$

The smallest amount of energy the electron can have corresponds to the lowest quantum number, which is $n = 1$ for the ground state of the electron. Thus, Eqs. 39-4 and 39-7 give us

$$E_1 = \left(\frac{h^2}{8mL^2}\right)n^2 = \left(6.031 \times 10^{-18} \mathrm{J}\right)\left(1^2\right)$$

$$\approx 6.03 \times 10^{-18} \mathrm{J} = 37.7 \mathrm{eV}. \qquad \text{(Answer)}$$

(b) How much energy must be transferred to the electron if it is to make a quantum jump from its ground state to its second excited state?

KEY IDEA

First a caution: Note that, from Fig. 39-3, the *second* excited state corresponds to the *third* energy level, with quantum number $n = 3$. Then if the electron is to jump from the $n = 1$ level to the $n = 3$ level, the required change in its energy is, from Eq. 39-5,

$$\Delta E_{31} = E_3 - E_1. \tag{39-8}$$

Upward jump:

The energies E_3 and E_1 depend on the quantum number n, according to Eq. 39-4. Therefore, substituting that equation into Eq. 39-8 for energies E_3 and E_1 and using Eq. 39-7 lead to

$$\Delta E_{31} = \left(\frac{h^2}{8mL^2}\right)(3)^2 - \left(\frac{h^2}{8mL^2}\right)\left(1^2\right)$$

$$= \frac{h^2}{8mL^2}\left(3^2 - 1^2\right) \qquad \text{(Answer)}$$

$$= \left(6.031 \times 10^{-18} \mathrm{J}\right)(8)$$

$$= 4.83 \times 10^{-17} \mathrm{J} = 301 \mathrm{eV}.$$

(c) If the electron gains the energy for the jump from energy level *E1* to energy level E_1 by absorbing light, what light wavelength is required?

KEY IDEAS

(1) If light is to transfer energy to the electron, the transfer must be by photon absorption.
(2) The photon's energy must equal the energy difference ΔE between the initial energy

level of the electron and a higher level, according to Eq. 39-6 ($hf = \Delta E$). Otherwise, a photon *cannot* be absorbed.

Wavelength:

Substituting c/λ for f, we can rewrite Eq. 39-6 as

$$\lambda = \frac{hc}{\Delta E}. \qquad (39\text{-}9)$$

For the energy difference ΔE_{31} we found in (b), this equation gives us

$$\begin{aligned}\lambda &= \frac{hc}{\Delta E_{31}} \\ &= \frac{\left(6.63 \times 10^{-34}\,\mathrm{J \cdot s}\right)\left(2.998 \times 10^{8}\,\mathrm{m/s}\right)}{4.83 \times 10^{-17}\,\mathrm{J}} \\ &= 4.12 \times 10^{-9}\,\mathrm{m}.\end{aligned} \qquad \text{(Answer)}$$

(d) Once the electron has been excited to the second excited state, what wavelengths of light can it emit by de-excitation?

KEY IDEAS

1. The electron tends to de-excite, rather than remain in an excited state, until it reaches the ground state ($n = 1$).
2. If the electron is to de-excite, it must lose just enough energy to jump to a lower energy level.
3. If it is to lose energy by emitting light, then the loss of energy must be by emission of a photon.

Downward jumps:

Starting in the second excited state (at the $n = 3$ level), the electron can reach the ground state ($n = 1$) by *either* making a quantum jump directly to the ground-state energy level (Fig. 39-5*a*) or by making two *separate* jumps by way of the $n = 2$ level (Figs. 39-5*b* and 39-5*c*).

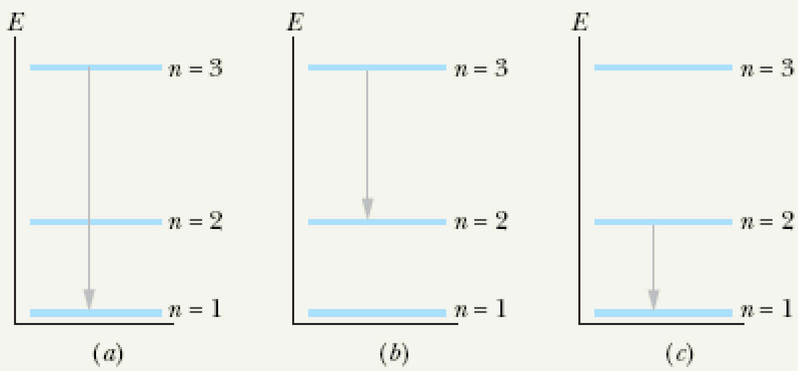


Figure 39-5 De-excitation from the second excited state to the ground state either directly (*a*) or via the first excited state (*b*, *c*).

The direct jump involves the same energy difference ΔE_{31} we found in (c). Then the wavelength is the same as we calculated in (c)—except now the wavelength is for light that is emitted, not absorbed. Thus, the electron can jump directly to the ground state by emitting light of wavelength

$$\lambda = 4.12 \times 10^{-9} \text{m}. \qquad \text{(Answer)}$$

Following the procedure of part (b), you can show that the energy differences for the jumps of Figs. 39-5*b* and 39-5*c* are

$$\Delta E_{32} = 3.016 \times 10^{-17} \text{J} \quad \text{and} \quad \Delta E_{21} = 1.809 \times 10^{-17} \text{J}.$$

From Eq. 39-9, we then find that the wavelength of the light emitted in the first of these jumps (from $n = 3$ to $n = 2$) is

$$\lambda = 6.60 \times 10^{-9} \text{m}, \qquad \text{(Answer)}$$

and the wavelength of the light emitted in the second of these jumps (from $n = 2$ to $n = 1$) is

$$\lambda = 1.10 \times 10^{-8} \text{m}. \qquad \text{(Answer)}$$

39-4 Wave Functions of a Trapped Electron

If we solve Schrödinger's equation for an electron trapped in a one-dimensional infinite potential well of width L, we find that the wave functions for the electron are given by

$$\psi_n(x) = A \sin\left(\frac{n\pi}{L}x\right), \qquad \text{for } n = 1, 2, 3, \ldots, \tag{39-10}$$

for $0 \le x \le L$ (the wave function is zero outside that range). We shall soon evaluate the amplitude constant A in this equation.

Note that the wave functions $\psi_n(x)$ have the same form as the displacement functions $y_n(x)$ for a standing wave on a string stretched between rigid supports (see Eq. 39-2). We can picture an electron trapped in a one-dimensional well between infinite-potential walls as being a standing matter wave.

Probability of Detection

The wave function $\psi_n(x)$ cannot be detected or directly measured in any way—we cannot simply look inside the well to see the wave the way we can see, say, a wave in a bathtub of water. All we can do is insert a probe of some kind to try to detect the electron. At the instant of detection, the electron would materialize at the point of detection, at some position along the *x* axis within the well.

If we repeated this detection procedure at many positions throughout the well, we would find that the probability of detecting the electron is related to the probe's position *x* in the well. In fact, they are related by the *probability density* $\psi_n^2(x)$. Recall from Section 38-7 that in general the probability that a particle can be detected in a specified infinitesimal volume centered on a specified point is proportional to $\left|\psi_n^2(x)\right|$. Here, with the electron trapped in a one-dimensional well, we are concerned only with detection of the electron along the *x* axis. Thus, the probability density $\psi_n^2(x)$ here is a probability per unit length along the *x* axis. (We can omit the absolute value sign here because $\psi_n(x)$ in Eq. 39-10 is a real quantity, not a complex one.) The probability $p(x)$ that an electron can be detected at position *x* within the well is

$$(\text{probability } p(x) \text{ of detection in width } dx \text{ centered on position } x) = \left(\text{probability density } \psi_n^2(x)\right.$$

or

$$p(x) = \psi_n^2(x)\, dx. \qquad (39\text{-}11)$$

From Eq. 39-10, we see that the probability density $\psi_n^2(x)$ is

$$\psi_n^2(x) = A^2 \sin^2\left(\frac{n\pi}{L} x\right), \qquad \text{for } n = 1, 2, 3, \ldots, \qquad (39\text{-}12)$$

for the range $0 \le x \le L$ (the probability density is zero outside that range). Figure 39-6 shows $\psi_n^2(x)$ for n = 1,2, 3, and 15 for an electron in an infinite well whose width L is 100 pm.

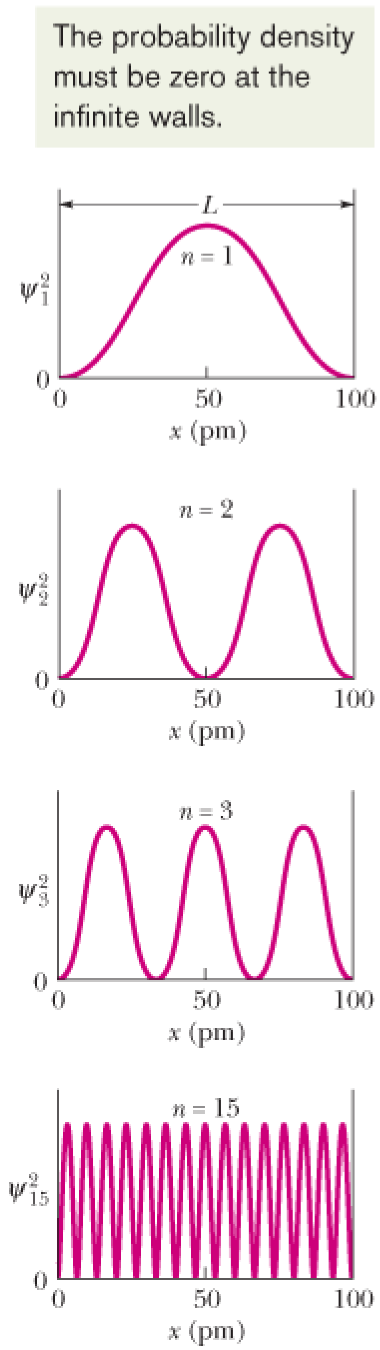


Figure 39-6

The probability density $\psi_n^2(x)$ for four states of an electron trapped in a one-dimensional infinite well; their quantum numbers are $n = 1, 2, 3$, and 15. The electron is most likely to be found where $\psi_n^2(x)$ is greatest and least likely to be found where $\psi_n^2(x)$ is least.

To find the probability that the electron can be detected in any finite section of the well—say, between point x_1 and point x_2—we must integrate $p(x)$ between those points. Thus, from Eqs. 39-11 and 39-12,

$$\begin{aligned}(\text{probability of detection between } x_1 \text{ and } x_2) &= \int_{x_1}^{x_2} p(x) \\ &= \int_{x_1}^{x_2} A^2 \sin^2\left(\frac{n\pi}{L}x\right)dx.\end{aligned} \tag{39-13}$$

If classical physics prevailed, we would expect the trapped electron to be detectable with equal probabilities in all parts of the well. From Fig. 39-6 we see that it is not. For example, inspection of that figure or of Eq. 39-12 shows that for the state with $n = 2$, the electron is most likely to be detected near $x = 25$ pm and $x = 75$ pm. It can be detected with near-zero probability near $x = 0$, $x = 50$ pm, and $x = 100$ pm.

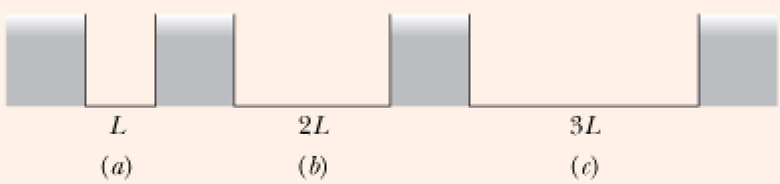
The case of $n = 15$ in Fig. 39-6 suggests that as n increases, the probability of detection becomes more and more uniform across the well. This result is an instance of a general principle called the **correspondence principle**:

At large enough quantum numbers, the predictions of quantum physics merge smoothly with those of classical physics.

This principle, first advanced by Danish physicist Niels Bohr, holds for all quantum predictions.

CHECKPOINT 2

The figure shows three infinite potential wells of widths L, $2L$, and $3L$; each contains an electron in the state for which $n = 10$. Rank the wells according to (a) the number of maxima for the probability density of the electron and (b) the energy of the electron, greatest first.



Answer:

(a) all tie; (b) *a*, *b*, *c*

Normalization

The product $\psi_n^2(x)dx$ gives the probability that an electron in an infinite well can be detected in the interval of the *x* axis that lies between *x* and *x* + *dx*. We know that the electron must be *somewhere* in the infinite well; so it must be true that

$$\int_{-\infty}^{+\infty} \psi_n^2(x)\, dx = 1 \qquad \text{(normalization equation)}, \tag{39-14}$$

because the probability 1 corresponds to certainty. Although the integral is taken over the entire *x* axis, only the region from *x* = 0 to *x* = *L* makes any contribution to the probability. Graphically, the integral in Eq. 39-14 represents the area under each of the plots of Fig. 39-6.

If we substitute $\psi_n^2(x)$ from Eq. 39-12 into Eq. 39-14, it is possible to assign a specific value to the amplitude constant *A* that appears in Eq. 39-12; namely, $A = \sqrt{2 / L}$. This process of using Eq. 39-14 to evaluate the amplitude of a wave function is called **normalizing** the wave function. The process applies to *all* one-dimensional wave functions.

CHECKPOINT 3

Each of the following particles is confined to an infinite well, and all four wells have the same width: (a) an electron, (b) a proton, (c) a deuteron, and (d) an alpha particle. Rank their zero-point energies, greatest first. The particles are listed in order of increasing mass.

Answer:

a, b, c, d

Zero-Point Energy

Substituting *n* = 1 in Eq. 39-4 defines the state of lowest energy for an electron in an infinite potential well, the ground state. That is the state the confined electron will occupy unless energy is supplied to it to raise it to an excited state.

The question arises: Why can't we include *n* = 0 among the possibilities listed for *n* in Eq. 39-4? Putting *n* = 0 in this equation would indeed yield a ground-state energy of zero. However, putting *n* = 0 in Eq. 39-12 would also yield $\psi_n^2(x) = 0$ for all *x*, which we can interpret only to mean that there is no electron in the well. We know that there is; so *n* = 0 is not a possible quantum number.

It is an important conclusion of quantum physics that confined systems cannot exist in states with zero energy. They must always have a certain minimum energy called the **zero-point energy**.

We can make the zero-point energy as small as we like by making the infinite well wider—that is, by increasing *L* in Eq. 39-4 for *n* = 1. In the limit as $L \to \infty$, the zero-point energy E_1 approaches zero. In this limit, however, with an infinitely wide well, the electron is a free particle, no longer confined in the x direction. Also, because the energy of a free particle is not quantized, that energy can have any value, including zero. Only a confined particle must have a finite zero-point energy and can never be at rest.

Detection probability in a 1D infinite potential well

(a) A ground-state electron is trapped in the one-dimensional infinite potential well of Fig. 39-2, with width L = 100 pm.

What is the probability that the electron can be detected in the left one-third of the well (x_1 = 0 to $x_2 = L/3$)?

KEY IDEAS

(1) If we probe the left one-third of the well, there is no guarantee that we will detect the electron. However, we can calculate the probability of detecting it with the integral of Eq. 39-13. (2) The probability very much depends on which state the electron is in—that is, the value of quantum number n.

Calculations:

Because here the electron is in the ground state, we set n = 1 in Eq. 39-13. We also set the limits of integration as the positions $X_1 = 0$ and $x_2 = L/3$ and set the amplitude constant A as $\sqrt{2/L}$(so that the wave function is normalized). We then see that

$$\text{(probability of detection in left one-third)} = \int_0^{L/3} \frac{2}{L}\sin^2\left(\frac{1\pi}{L}x\right)dx.$$

We could find this probability by substituting 100×10^{-12} m for L and then using a graphing calculator or a computer math package to evaluate the integral. Here, however, we shall evaluate the integral "by hand." First we switch to a new integration variable y:

$$y = \frac{\pi}{L}x \quad \text{and} \quad dx = \frac{L}{\pi}dy.$$

From the first of these equations, we find the new limits of integration to be $y_1 = 0$ for $x_1 = 0$ and $y_2 = \pi/3$ for $x_2 = L/3$. We then must evaluate

$$\text{probability} = \left(\frac{2}{L}\right)\left(\frac{L}{\pi}\right)\int_0^{\pi/3}\left(\sin^2 y\right)dy.$$

Using integral 11 in Appendix E, we then find

$$\text{probability} = \frac{2}{L}\left(\frac{L}{\pi}\right)\left(\frac{y}{2} - \frac{\sin 2y}{4}\right)_0^{\pi/3} = 0.20.$$

Thus, we have

$$\text{(probability of detection in left one-third)} = 0.20. \quad \text{(Answer)}$$

That is, if we repeatedly probe the left one-third of the well, then on average we can detect the electron with 20% of the probes.

(b) What is the probability that the electron can be detected in the middle one-third of the well?

Reasoning: We now know that the probability of detection in the left one-third of the well is 0.20. By symmetry, the probability of detection in the right one-third of the well is also 0.20. Because the electron is certainly in the well, the probability of detection in the entire well is 1. Thus, the probability of detection in the middle one-third of the well is

$$\begin{aligned}\text{(probability of detection in middle one-third)} &= 1 - 0.20 - 0.20 \\ &= 0.60.\end{aligned} \quad \text{(Answer)}$$

Normalizing wave functions in a 1D infinite potential well

Evaluate the amplitude constant *A* in Eq. 39-10 for an infinite potential well extending from *x = 0 to x = L*.

KEY IDEA

The wave functions of Eq. 39-10 must satisfy the normalization requirement of Eq. 39-14, which states that the probability that the electron can be detected somewhere along the *x* axis is 1.

Calculations:

Substituting Eq. 39-10 into Eq. 39-14 and taking the constant *A* outside the integral yield

$$A^2 \int_0^L \sin^2\left(\frac{n\pi}{L}x\right) dx = 1. \quad (39\text{-}15)$$

We have changed the limits of the integral from $-\infty$ and $+\infty$ to 0 and *L* because the wave function is zero outside these new limits (so there's no need to integrate out there).

We can simplify the indicated integration by changing the variable from *x* to the dimensionless variable *y*, where

$$y = \frac{n\pi}{L}x, \quad (39\text{-}16)$$

hence

$$dx = \frac{L}{n\pi}dy.$$

When we change the variable, we must also change the integration limits (again). Equation 39-16 tells us that $y = 0$ when $x = 0$ and that $y = n\pi$ when $x = L$; thus 0 and $n\pi$ are our new limits. With all these substitutions, Eq. 39-15 becomes

$$A^2 \frac{L}{n\pi} \int_0^{n\pi} \left(\sin^2 y\right) dy = 1.$$

We can use integral 11 in Appendix E to evaluate the integral, obtaining the equation

$$\frac{A^2 L}{n\pi} \left[\frac{y}{2} - \frac{\sin 2y}{4} \right]_0^{n\pi} = 1.$$

Evaluating at the limits yields

$$\frac{A^2 L}{n\pi} \frac{n\pi}{2} = 1;$$

thus

$$A = \sqrt{\frac{2}{L}}. \qquad \text{(Answer)} \qquad (39\text{-}17)$$

This result tells us that the dimension for A^2, and thus for $\psi_n^2(x)$, is an inverse length. This is appropriate because the probability density of Eq. 39-12 is a probability *per unit length*.

39-5 An Electron in a Finite Well

A potential energy well of infinite depth is an idealization. Figure 39-7 shows a realizable potential energy well—one in which the potential energy of an electron outside the well is not infinitely great but has a finite positive value U_0, called the **well depth**. The analogy between waves on a stretched string and matter waves fails us for wells of finite depth because we can no longer be sure that matter wave nodes exist at $x = 0$ and at $x = L$. (As we shall see, they don't.)

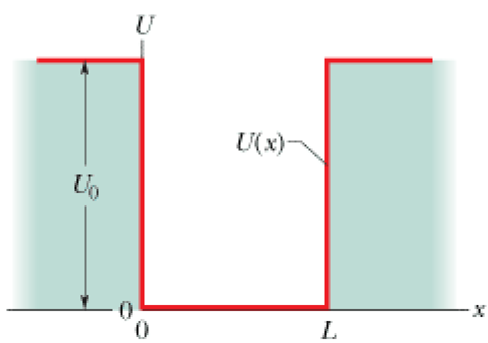


Figure 39-7 A *finite* potential energy well. The depth of the well is U_0 and its width is L. As in the

infinite potential well of Fig. 39-2, the motion of the trapped electron is restricted to the *x* direction.

To find the wave functions describing the quantum states of an electron in the finite well of Fig. 39-7, we *must* resort to Schrödinger's equation, the basic equation of quantum physics. From Section 38-7 recall that, for motion in one dimension, we use Schrödinger's equation in the form of Eq. 38-15:

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}[E - U(x)]\psi = 0. \tag{39-18}$$

Rather than attempting to solve this equation for the finite well, we simply state the results for particular numerical values of U_0 and L. Figure 39-8 shows three results as graphs of $\psi_n^2(x)$, the probability density, for a well with U_0 = 450 eV and L = 100 pm.

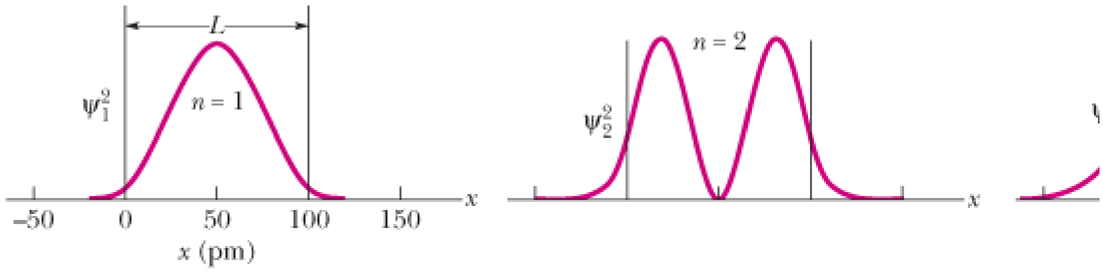


Figure 39-8
The first three probability densities $\psi_n^2(x)$ for an electron confined to a finite potential well of depth U_0 = 450 eV and width L = 100 pm. Only states n = 1,2,3, and 4 are allowed.

The probability density $\psi_n^2(x)$ for each graph in Fig. 39-8 satisfies Eq. 39-14, the normalization equation; so we know that the areas under all three probability density plots are numerically equal to 1.

If you compare Fig. 39-8 for a finite well with Fig. 39-6 for an infinite well, you will see one striking difference: For a finite well, the electron matter wave penetrates the walls of the well—into a region in which Newtonian mechanics says the electron cannot exist. This penetration should not be surprising because we saw in Section 38-9 that an electron can tunnel through a potential energy barrier. “Leaking” into the walls of a finite potential energy well is a similar phenomenon. From the plots of ψ^2 in Fig. 39-8, we see that the leakage is greater for greater values of quantum number n.

Because a matter wave *does* leak into the walls of a finite well, the wavelength λ for any given quantum state is greater when the electron is trapped in a finite well than when it is trapped in an infinite well. Equation 39-3 $h/\sqrt{2mE}$ then tells us that the energy E for an electron in any given state is less in the finite well than in the infinite well.

That fact allows us to approximate the energy-level diagram for an electron trapped in a finite well. As an example, we can approximate the diagram for the finite well of Fig. 39-8, which has width L = 100 pm and depth U_0 = 450 eV. The energy-level diagram for an *infinite* well of that width is shown in Fig. 39-3. First we remove the portion of Fig. 39-3 above 450 eV. Then we shift the remaining four energy levels down, shifting the level for n = 4 the most because the wave leakage into the walls is greatest for n = 4. The result is approximately the energy-level diagram for the finite well. The actual diagram is Fig. 39-9.

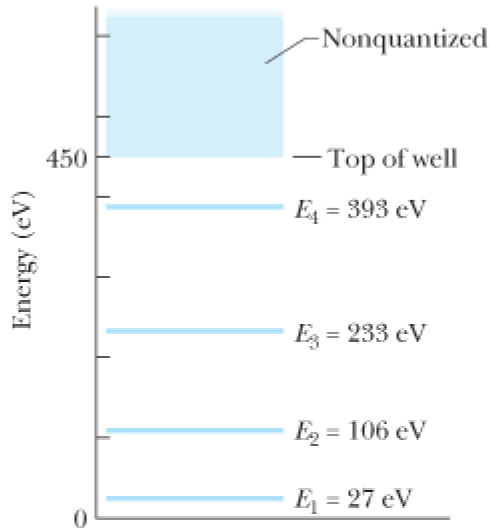


Figure 39-9 The energy-level diagram corresponding to the probability densities of Fig. 39-8. If an electron is trapped in the finite potential well, it can have only the energies corresponding to n = 1,2,3, and 4. If it has an energy of 450 eV or greater, it is not trapped and its energy is not quantized.

In that figure, an electron with an energy greater than U_0 (= 450 eV) has too much energy to be trapped in the finite well. Thus, it is not confined, and its energy is not quantized; that is, its energy is not restricted to certain values. To reach this *nonquantized* portion of the energy-level diagram and thus to be free, a trapped electron must somehow obtain enough energy to have a mechanical energy of 450 eV or greater.

Electron escaping from a finite potential well

(a) Suppose a finite well with U_0 = 450 eV and L = 100 pm confines a single electron in its ground state.

What wavelength of light is needed to barely free the electron from the potential well if the electron absorbs a single photon from the light?

KEY IDEA

For the electron to escape from the potential well, it must receive enough energy to put it into the nonquantized energy region of Fig. 39-9. Thus, it must end up with an energy of at least U_0(= 450 eV).

Barely escaping:

The electron is initially in its ground state, with an energy of E_1 = 27 eV. So, to barely become free, it must receive an energy of

$$U_0 - E_1 = 450\text{eV} - 27\text{eV} = 423\text{eV}.$$

Thus the photon must have this much energy. From Eq. 39-6 ($hf = E_{high} - E_{low}$), with c/λ substituted for f, we write

$$\frac{hc}{\lambda} = U_0 - E_1,$$

from which we find

$$\begin{aligned}\lambda &= \frac{hc}{U_0 - E_1} \\ &= \frac{\left(6.63\times10^{-34}\,\mathrm{J\cdot s}\right)\left(3.00\times10^{8}\,\mathrm{m/s}\right)}{(423\,\mathrm{eV})\left(1.60\times10^{-9}\,\mathrm{J/eV}\right)} \\ &= 2.94\times10^{-9}\,\mathrm{m} = 2.94\,\mathrm{nm}.\end{aligned}$$ (Answer)

Thus, if λ = 2.94 nm, the electron just barely escapes.

(b) Can the ground-state electron absorb light with λ = 2.00 nm? If so, what then is the electron's energy?

KEY IDEAS

1. In (a) we found that light of 2.94 nm will just barely free the electron from the potential well.
2. We are now considering light with a shorter wavelength of 2.00 nm and thus a greater energy per photon ($hf = hc/\lambda$).
3. Hence, the electron *can* absorb a photon of this light. The energy transfer will not only free the electron but will also provide it with more kinetic energy. Further, because the electron is then no longer trapped, its energy is not quantized.

More than escaping:

The energy transferred to the electron is the photon energy:

$$\begin{aligned}hf &= h\frac{c}{\lambda} = \frac{\left(6.63\times10^{-34}\,\mathrm{J\cdot s}\right)\left(3.00\times10^{8}\,\mathrm{m/s}\right)}{2.00\times10^{-9}\,\mathrm{m}} \\ &= 9.95\times10^{-17}\,\mathrm{J} = 622\,\mathrm{eV}.\end{aligned}$$

From (a), the energy required to just barely free the electron from the potential well is $U_0 - E_1$ (= 423 eV). The remainder of the 622 eV goes to kinetic energy. Thus, the kinetic energy of the freed electron is

$$\begin{aligned}K &= hf - (U_0 - E_1) \\ &= 622\,\mathrm{eV} - 423\,\mathrm{eV} = 199\,\mathrm{eV}.\end{aligned}$$ (Answer)

39-6 More Electron Traps

Here we discuss three types of artificial electron traps.

Nanocrystallites

Perhaps the most direct way to construct a potential energy well in the laboratory is to prepare a sample of a semiconducting material in the form of a powder whose granules are small—in the nanometer range—and of uniform size. Each such granule—each **nanocrystallite**—acts as a potential well for the electrons trapped within it.

Equation 39-4 ($E = (h^2/8mL^2)n^2$) shows that we can increase the energy-level values of an electron trapped in an infinite well by reducing the width L of the well. This would also shift the photon energies that the well can absorb to higher values and thus shift the corresponding wavelengths to shorter values.

These general results are also true for a well formed by a nanocrystallite. A given nanocrystallite can absorb photons with an energy above a certain threshold energy E_t (= hf_t) and thus wavelengths below a corresponding threshold wavelength

$$\lambda_t = \frac{c}{f_1} = \frac{ch}{E_t}.$$

Light with any wavelength longer than λ_t is scattered by the nanocrystallite instead of being absorbed. The color we attribute to the nanocrystallite is then determined by the wavelength composition of the scattered light we intercept.

If we reduce the size of the nanocrystallite, the value of E_t is increased, the value of λ_t is decreased, and the light that is scattered to us changes in its wavelength composition. Thus, the color we attribute to the nanocrystallite changes. As an example, Fig. 39-10 shows two samples of the semiconductor cadmium selenide, each consisting of a powder of nanocrystallites of uniform size. The lower sample scatters light at the red end of the spectrum. The upper sample differs from the lower sample *only* in that the upper sample is composed of smaller nanocrystallites. For this reason its threshold energy E_t is greater and, from above, its threshold wavelength λ_t is shorter, in the green range of visible light. Thus, the sample now scatters both red and yellow. Because the yellow component happens to be brighter, the sample's color is now dominated by the yellow. The striking contrast in color between the two samples is compelling evidence of the quantization of the energies of trapped electrons and the dependence of these energies on the size of the electron trap.

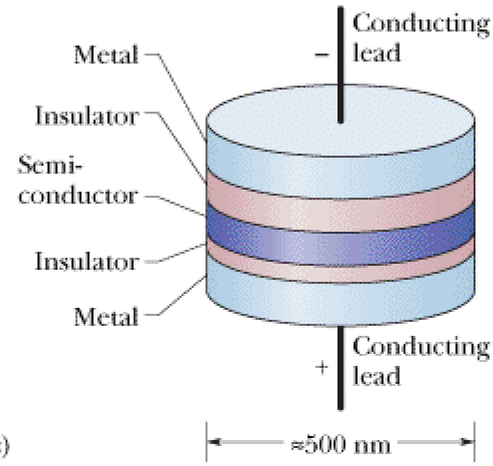
Figure 39-10 Two samples of powdered cadmium selenide, a semiconductor, differing only in the size of their granules. Each granule serves as an electron trap. The lower sample has the larger granules and consequently the smaller spacing between energy levels and the lower photon energy threshold for the absorption of light. Light not absorbed is scattered, causing the sample to scatter light of greater wavelength and appear red. The upper sample, because of its smaller granules, and consequently its larger level spacing and its larger energy threshold for absorption, appears yellow.
(From Scientific American, January 1993, page 122. Reproduced with permission of Michael Steigerwald, Bell Labs-Lucent Technologies)

Quantum Dots

The highly developed techniques used to fabricate computer chips can be used to construct, atom by atom, individual potential energy wells that behave, in many respects, like artificial atoms. These **quantum dots**, as they are usually called, have promising applications in electron optics and computer technology.

In one such arrangement, a “sandwich” is fabricated in which a thin layer of a semiconducting material, shown in purple in Fig. 39-11*a*, is deposited between two insulating layers, one of which is much thinner than the other. Metal end caps with conducting leads are added at both ends. The materials are chosen to ensure that the potential energy of an electron in the central layer is less than it is in the two insulating layers, causing the central layer to act as a potential energy well. Figure 39-11*b* is a photograph of an actual quantum dot; the well in which individual electrons can be trapped is the purple region.

(a)



(b)

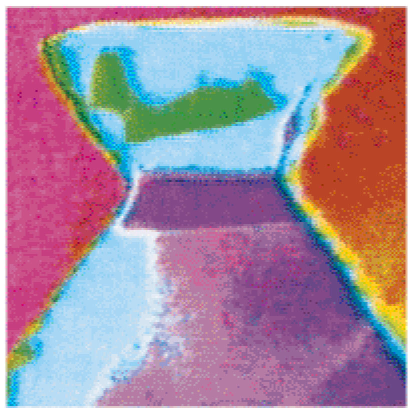


Figure 39-11 A quantum dot, or "artificial atom." (*a*) A central semiconducting layer forms a potential energy well in which electrons are trapped. The lower insulating layer is thin enough to allow electrons to be added to or removed from the central layer by barrier tunneling if an appropriate voltage is applied between the leads. (*b*) A photograph of an actual quantum dot. The central purple band is the electron confinement region.
(From Scientific American, September 1995, page 67. Image reproduced with permission of H. Temkin, Texas Tech University)

The lower (but not the upper) insulating layer in Fig. 39-11*a* is thin enough to permit electrons to tunnel through it if an appropriate potential difference is applied between the leads. In this way the number of electrons confined to the well can be controlled. The arrangement does indeed behave like an artificial atom with the property that the number of electrons it contains can be controlled. Quantum dots can be constructed in two-dimensional arrays that could well form the basis for computing systems of great speed and storage capacity.

Quantum Corrals

When a scanning tunneling microscope (described in Section 38-9) is in operation, its tip exerts a small force on isolated atoms that may be located on an otherwise smooth surface. By careful manipulation of the position of the tip, such isolated atoms can be "dragged" across the surface and deposited at another location. Using this technique, scientists at IBM's Almaden Research Center moved iron atoms across a carefully prepared copper surface, forming the atoms into a circle (Fig. 39-12), which they named a **quantum corral**. Each iron atom in the circle is nestled in a hollow in the copper surface, equidistant from three nearest-neighbor copper atoms. The corral was fabricated at a low temperature (about 4 K) to minimize the tendency of the iron atoms to move randomly about on the surface because of their thermal energy.

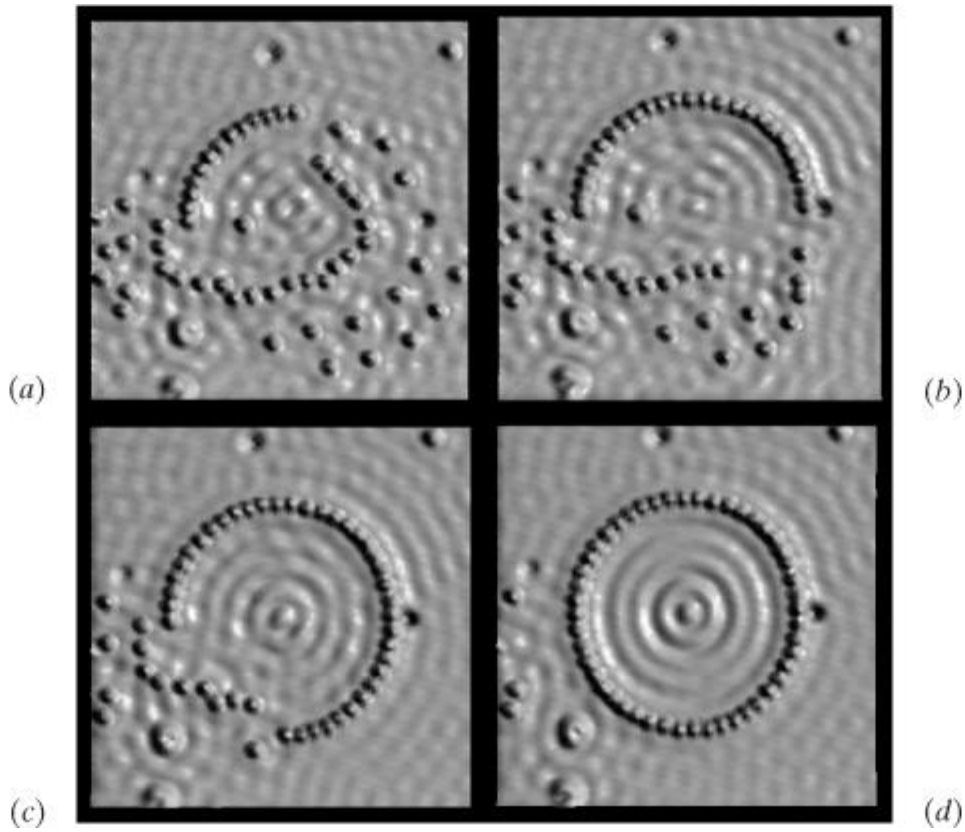


Figure 39-12 A quantum corral during four stages of construction. Note the appearance of ripples caused by electrons trapped in the corral when it is almost complete.
(Courtesy of International Business Machines Corporation, Almaden Research Center)

The ripples within the corral are due to matter waves associated with electrons that can move over the copper surface but are largely trapped in the potential well of the corral. The dimensions of the ripples are in excellent agreement with the predictions of quantum theory.

39-7 Two- and Three-Dimensional Electron Traps

In the next section, we shall discuss the hydrogen atom as being a three-dimensional finite potential well. As a warm-up for the hydrogen atom, let us extend our discussion of infinite potential wells to two and three dimensions.

Rectangular Corral

Figure 39-13 shows the rectangular area to which an electron can be confined by the two-dimensional version of Fig. 39-2—a two-dimensional infinite potential well of widths L_x and L_y that forms a rectangular corral. The corral might be on the surface of a body that somehow prevents the electron from moving parallel to the *z* axis and thus from leaving the surface. You have to imagine infinite potential energy functions (like $U(x)$ in Fig. 39-2) along each side of the corral, keeping the electron within the corral.

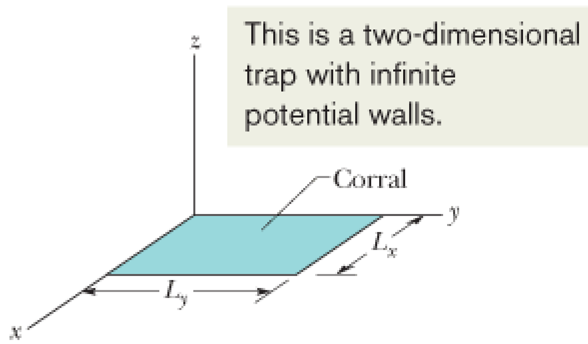


Figure 39-13 A rectangular corral—a two-dimensional version of the infinite potential well of Fig. 39-2—with widths L_x and L_y.

Solution of Schrödinger's equation for the rectangular corral of Fig. 39-13 shows that, for the electron to be trapped, its matter wave must fit into each of the two widths separately, just as the matter wave of a trapped electron must fit into a one-dimensional infinite well. This means the wave is separately quantized in width L_x and in width L_y. Let n_x be the quantum number for which the matter wave fits into width L_x, and let n_y be the quantum number for which the matter wave fits into width L_y. As with a one-dimensional potential well, these quantum numbers can be only positive integers. We can extend Eqs. 39-10 and 39-17 to write the normalized wave function as

$$\psi_{nx,ny} = \sqrt{\frac{2}{L_x}}\sin\left(\frac{n_x\pi}{L}x\right)\sqrt{\frac{2}{L_y}}\sin\left(\frac{n_y\pi}{L}y\right). \tag{39-19}$$

The energy of the electron depends on both quantum numbers and is the sum of the energy the electron would have if it were confined along the *x* axis alone and the energy it would have if it were confined along the *y* axis alone. From Eq. 39-4, we can write this sum as

$$E_{nx,ny} = \left(\frac{h^2}{8mL_x^2}\right)n_x^2 + \left(\frac{h^2}{8mL_y^2}\right)n_y^2 = \frac{h^2}{8m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2}\right). \tag{39-20}$$

Excitation of the electron by photon absorption and de-excitation of the electron by photon emission have the same requirements as for one-dimensional traps. Now, however, two quantum numbers (n_x and n_y) are involved. Because of that, different states might have the same energy; such states and their energy levels are said to be *degenerate*.

Rectangular Box

An electron can also be trapped in a three-dimensional infinite potential well—a *box*. If the box is rectangular as in Fig. 39-14, then Schrödinger's equation shows us that we can write the energy of the electron as

$$E_{nx,ny,nz} = \frac{h^2}{8m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right). \tag{39-21}$$

Here n_z is a third quantum number, for fitting the matter wave into width L_z.

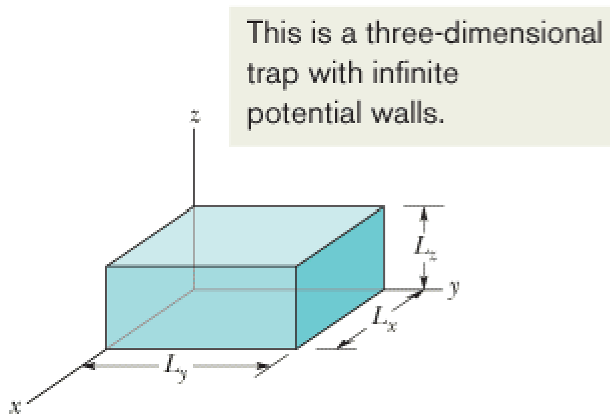


Figure 39-14 A rectangular box—a three-dimensional version of the infinite potential well of Fig. 39-2—with widths L_x, L_y, and L_z.

CHECKPOINT 4

In the notation of Eq. 39-20, is $E_{0,0}$, $E_{1,0}$, $E_{0,1}$, or $E_{1,1}$ the ground-state energy of an electron in a (two-dimensional) rectangular corral?

Answer:

$E_{1,1}$ (neither n_x nor n_y can be zero)

Energy levels in a 2D infinite potential well

(a) An electron is trapped in a square corral that is a two-dimensional infinite potential well (Fig. 39-13) with widths $L_x = L_y$.

(a) Find the energies of the lowest five possible energy levels for this trapped electron, and construct the corresponding energy-level diagram.

KEY IDEA

Because the electron is trapped in a two-dimensional well that is rectangular, the electron's energy depends on two quantum numbers, n_x and n_y, according to Eq. 39-20.

Energy levels:

Because the well here is square, we can let the widths be $L_x = L_y = L$. Then Eq. 39-20 simplifies to

$$E_{nx,ny} = \frac{h^2}{8mL^2}\left(n_x^2 + n_y^2\right). \tag{39-22}$$

The lowest energy states correspond to low values of the quantum numbers n_x and n_y, which are the positive integers 1, 2,…, ∞. Substituting those integers for n_x and n_y in Eq. 39-22, starting with the lowest value 1, we can obtain the energy values as listed in Table 39-1. There we can see that several of the pairs of quantum numbers (n_x, n_y) give the same energy. For example, the (1,2) and (2,1) states both have an energy of $5(h^2/8mL^2)$. Each such pair is associated with degenerate energy levels. Note also that, perhaps surprisingly, the (4,1) and (1,4) states have less energy than the (3,3) state.

Table 39-1 Energy Levels

n_x	n_y	Energy[a]	n_x	n_y	Energy[a]
1	3	10	2	4	20
3	1	10	4	2	20
2	2	8	3	3	18
1	2	5	1	4	17
2	1	5	4	1	17
1	1	2	2	3	13
			3	2	13

a)In multiples of $h^2/8mL^2$.

From Table 39-1 (carefully keeping track of degenerate levels), we can construct the energy-level diagram of Fig. 39-15.

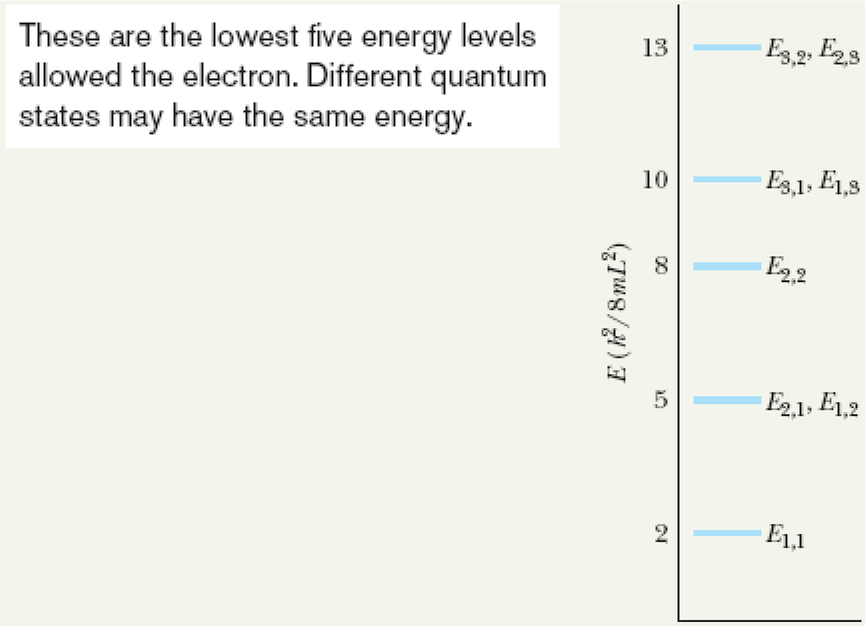


Figure 39-15 Energy-level diagram for an electron trapped in a square corral.

(b)(b) As a multiple of $h^2/8mL^2$, what is the energy difference between the ground state and the third excited state?

Energy difference: From Fig. 39-15, we see that the ground state is the (1,1) state, with an energy of $2(h^2/8mL^2)$. We also see that the third excited state (the third state up from the ground state in the energy-level diagram) is the degenerate (1, 3) and (3,1) states, with an energy of $10(h^2/8mL^2)$. Thus, the difference ΔE between these two states is

$$\Delta E = 10\left(\frac{h^2}{8mL^2}\right) - 2\left(\frac{h^2}{8mL^2}\right) = 8\left(\frac{h^2}{8mL^2}\right). \qquad \text{(Answer)}$$

39-8 The Bohr Model of the Hydrogen Atom

We now move from artificial and fictitious electron traps to natural ones—atoms. In this chapter we focus on the simplest example, a hydrogen atom, which contains an electron that is trapped to be near the proton, which forms the atom's central nucleus. Here we do not consider anything about the nucleus. Rather, we simply use the fact that the negatively charged electron is attracted by the Coulomb force to the positively charged proton. Because the proton mass is much greater than the electron mass, we assume that the proton is fixed in place.

We have now discussed at length that confinement of an electron means that the electron's energy E is quantized and thus so is any change ΔE in its energy. In this section we want to calculate the quantized energies of the electron in a hydrogen atom. Before we apply the wave approach we used in infinite and finite potential wells, however, let's explore the hydrogen atom at the dawn of quantum physics, when physicists first discovered that atoms are quantized systems.

By the early 1900s, scientists understood that matter came in tiny pieces called atoms and that an atom of hydrogen contained positive charge $+e$ at its center and negative charge $-e$ (an electron) outside that center. However, no one understood why the electrical attraction between the electron and the positive charge did not simply cause the two to collapse together.

One clue came from the fact that a hydrogen atom cannot emit and absorb all wavelengths of visible light. Rather, it can emit and absorb only four particular wavelengths from the visible range. By guesswork, Johann Balmer devised a formula that gave those wavelengths:

$$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right), \quad \text{for } 3, 4, 5, \text{ and } 6, \tag{39-23}$$

where R is a constant. However, no one knew why this formula gave the right wavelengths or why no other visible wavelengths are emitted and absorbed.

No one knew until 1913, when Bohr saw Balmer's equation and quickly realized that he could derive it if he made several bold (completely unjustified) assumptions: (1) The electron in a hydrogen atom orbits the nucleus in a circle much like Earth orbits the Sun (Fig. 39-16*a*). (2) The magnitude of the angular momentum $\vec{L}$ of the electron in its orbit is restricted to the values

$$L = n\hbar, \quad \text{for } 1, 2, 3, \ldots, \tag{39-24}$$

where $\hbar$ (h-bar) is $h/2\pi$ and n is a quantum number. Let's see the results.

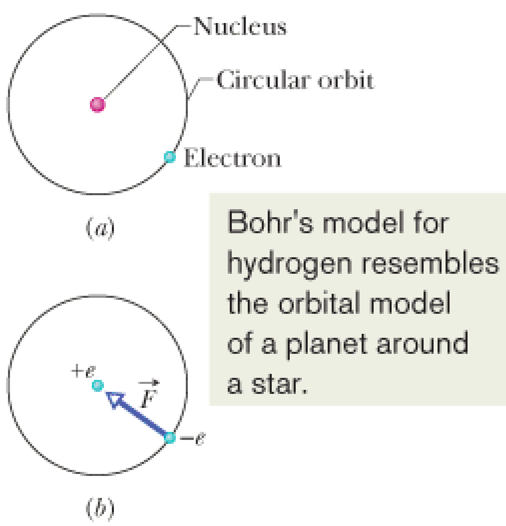


Figure 39-16(*a*) Circular orbit of an electron in the Bohr model of the hydrogen atom. (*b*) The Coulomb force $\vec{F}$ on the electron is directed radially inward toward the nucleus.

The Orbital Radius Is Quantized in the Bohr Model

Let's examine the orbital motion of the electron in the Bohr model. The force holding the electron in an orbit of radius r is the Coulomb force. From Eq. 21-1, we know that the magnitude of this force is

$$F = k\frac{|q_1||q_2|}{r^2},$$

with $k = 1/4\pi\varepsilon_0$. Here q_1 is the charge - e of the electron and q_2 is the charge + e of the nucleus (the proton). The electron's acceleration is the centripetal acceleration, with a magnitude given by $a = v^2/r$, where v is the electron's speed. Both force $\vec{F}$ and acceleration $\vec{a}$ are radially inward (the negative direction on a radial axis), toward the nucleus (Fig. 39-16*b*). Thus, we can write Newton's second law ($F = ma$) for a radial axis as

$$-\frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = m\left(-\frac{v^2}{r}\right), \tag{39-25}$$

where m is the electron mass.

We next introduce quantization by using Bohr's assumption expressed in Eq. 39-24. From Eq. 11-19, the magnitude ℓ, of the angular momentum of a particle of mass m and speed v moving in a circle of radius r is $\ell = rmv$ sin , where (the angle between $\vec{r}$ and $\vec{v}$) is 90°. Replacing L in Eq. 39-24 with rmv sin 90° gives us

$$rmv = n\hbar,$$

or

$$v = \frac{n\hbar}{rm}. \tag{39-26}$$

Substituting this equation into Eq. 39-25, replacing $\hbar$ with $h/2ir$, and rearranging, we find

$$r = \frac{h^2\epsilon_0}{\pi me^2}n^2, \quad \text{for } n = 1, 2, 3, \ldots. \tag{39-27}$$

We can rewrite this as

$$r = an^2, \quad \text{for } n = 1, 2, 3, \ldots, \tag{39-28}$$

where

$$a = \frac{h^2\epsilon_0}{\pi me^2} = 5.291\,772 \times 10^{-11}\,\text{m} \approx 52.92\,\text{pm}. \tag{39-29}$$

These last three equations tell us that, in the *Bohr model of the hydrogen atom*, the electron's orbital radius r is quantized and the smallest possible orbital radius (for $n = 1$) is a, which is now called the *Bohr radius*. According to the Bohr model, the electron cannot get any closer to the nucleus than orbital radius a, and that is why the attraction between electron and nucleus does not simply collapse them together.

Orbital Energy Is Quantized

Let's next find the energy of the hydrogen atom according to the Bohr model. The electron has kinetic energy $K = \frac{1}{2}mv^2$, and the electron-nucleus system has electric potential energy $U = q_1q_2/4\pi\varepsilon_0 r$ (Eq. 24-43). Again, let q_1 be the electron's charge - e and q_2 be the nuclear charge $+e$. Then the mechanical energy is

$$E = K + U$$
$$= \frac{1}{2}mv^2 + \left(-\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}\right). \tag{39-30}$$

Solving Eq. 39-25 for mv^2 and substituting the result in Eq. 39-30 lead to

$$E = -\frac{1}{8\pi\epsilon_0}\frac{e^2}{r}. \tag{39-31}$$

Next, replacing r with its equivalent from Eq. 39-27, we have

$$E_n = -\frac{me^4}{8\epsilon_0^2 h^2}\frac{1}{n^2}, \quad \text{for } n = 1, 2, 3 \ldots, \tag{39-32}$$

where the subscript n on E signals that we have now quantized the energy. Evaluating the constants in Eq. 39-32 gives us

$$E_n = -\frac{2.180 \times 10^{-18}\ \text{J}}{n^2} = -\frac{13.61\ \text{eV}}{n^2}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{39-33}$$

This equation tells us that the energy E_n of the hydrogen atom is quantized; that is, E_n is restricted by its dependence on the quantum number n. Because the nucleus is assumed to be fixed in place and only the electron has motion, we can assign the energy values of Eq. 39-33 either to the atom as a whole or to the electron alone.

Energy Changes

The energy of a hydrogen atom (or, equivalently, of its electron) changes when the atom emits or absorbs light. As we have seen several times since Eq. 39-6, emission and absorption involve a quantum of light according to

$$hf = \Delta E = E_{\text{high}} - E_{\text{low}}. \tag{39-34}$$

Let's make three changes to Eq. 39-34. On the left side, we substitute c/\ forf. On the right side, we use Eq. 39-32 twice to replace the energy terms. Then, with a simple rearrangement, we have

$$\frac{1}{\lambda} = -\frac{me^4}{8\epsilon_0^2 h^3 c}\left(\frac{1}{n_{\text{high}}^2} - \frac{1}{n_{\text{low}}^2}\right). \tag{39-35}$$

We can rewrite this as

$$\frac{1}{\lambda} = R\left(\frac{1}{n_{\text{low}}^2} - \frac{1}{n_{\text{high}}^2}\right), \tag{39-36}$$

in which

$$R = \frac{me^4}{8\epsilon_0^2 h^3 c} 1.097\,373 \times 10^7 \text{m}^{-1} \tag{39-37}$$

is now known as the *Rydberg constant*.

Compare Eq. 39-36 from the Bohr model with Eq. 39-23 from Balmer's work. In Eq. 39-36, if we replace n_{low} with 2 and then restrict n_{high} to be 3,4,5, and 6, we have Balmer's equation. This match was a triumph for Bohr and ushered in the quantum physics of atoms. The triumph was short-lived, however, because even though the Bohr model gives the correct emission and absorption wavelengths for the hydrogen atom, the model is *not* correct because the electron does *not* orbit the nucleus like a planet orbiting the Sun. Indeed, researchers found little success in extending the Bohr model to atoms more complicated than hydrogen. The reason for this lack of success is that an electron trapped in any atom is a matter wave confined to a potential well, and to find the resulting quantized energy values we must apply Schrödinger's equation to the electron.

39-9 Schrödinger's Equation and the Hydrogen Atom

The potential well of a hydrogen atom depends on the electrical potential energy function

$$U(r) = \frac{-e^2}{4\pi\epsilon_0 r}. \tag{39-38}$$

Because this well is three-dimensional, it is more complex than our previous one- and two-dimensional wells. Because this well is finite, it is more complex than the three-dimensional well of Fig. 39-14. Moreover, it does not have sharply defined walls. Rather, its walls vary in depth with radial distance r. Figure 39-17 is probably the best we can do in drawing the hydrogen potential well, but even that drawing takes much effort to interpret.

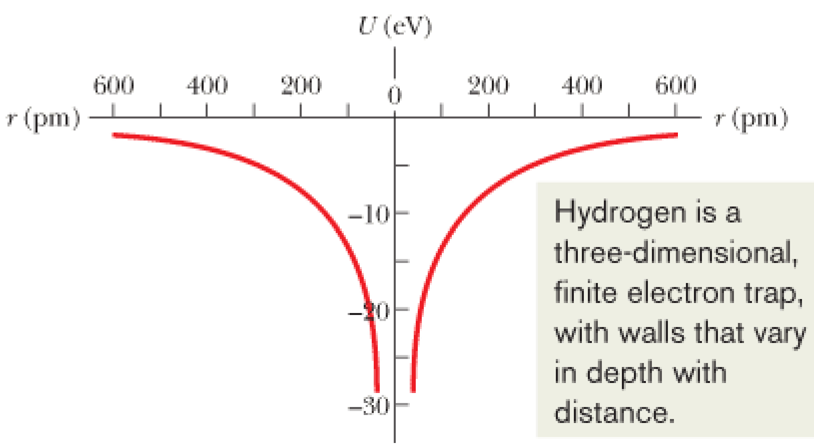


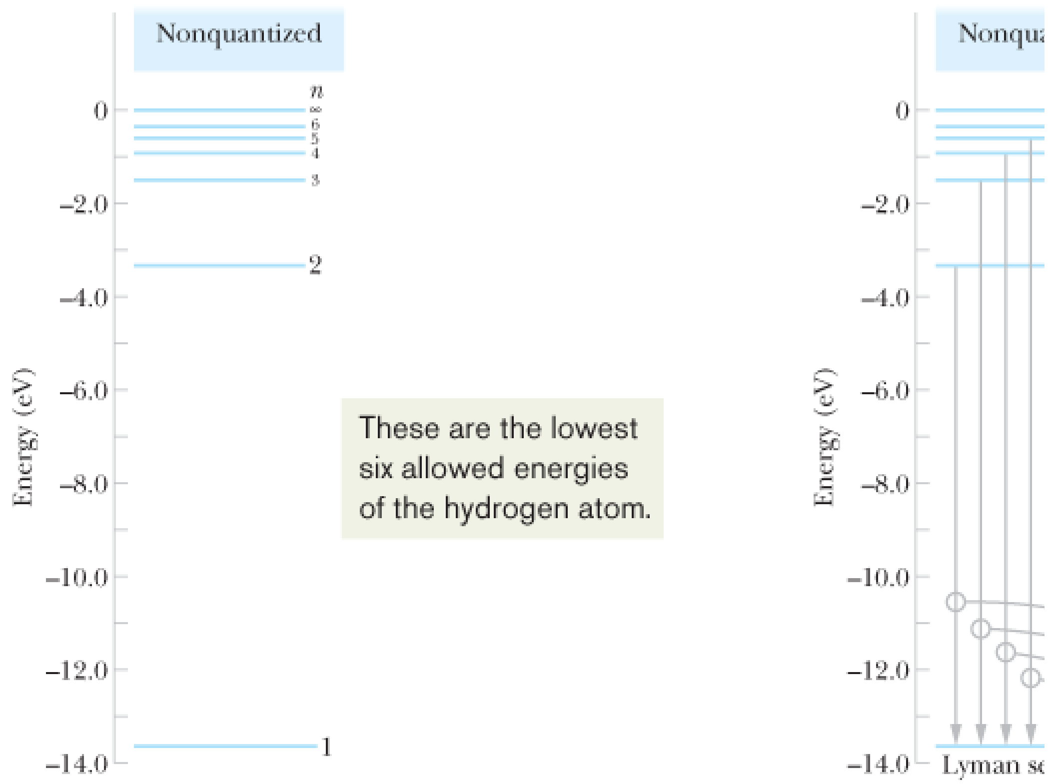
Figure 39-17 The potential energy U of a hydrogen atom as a function of the separation r between the electron and the central proton. The plot is shown twice (on the left and on the right) to suggest the three-dimensional spherically symmetric trap in which the electron is confined.

Energy Levels and Spectra of the Hydrogen Atom

Although we shall not do so here, we can apply Schrödinger's equation for an electron trapped in the potential well given by Eq. 39-38. In doing so, we would find that the energy values are quantized and that, amazingly, those values are given by Eq. 39-33 just as for the (incorrect) Bohr model. Thus, changes ΔE in energy due to emission or absorption of light are given by Eq. 39-34, and the wavelengths corresponding to ΔE are given by Eq. 39-36. Let's explore these results.

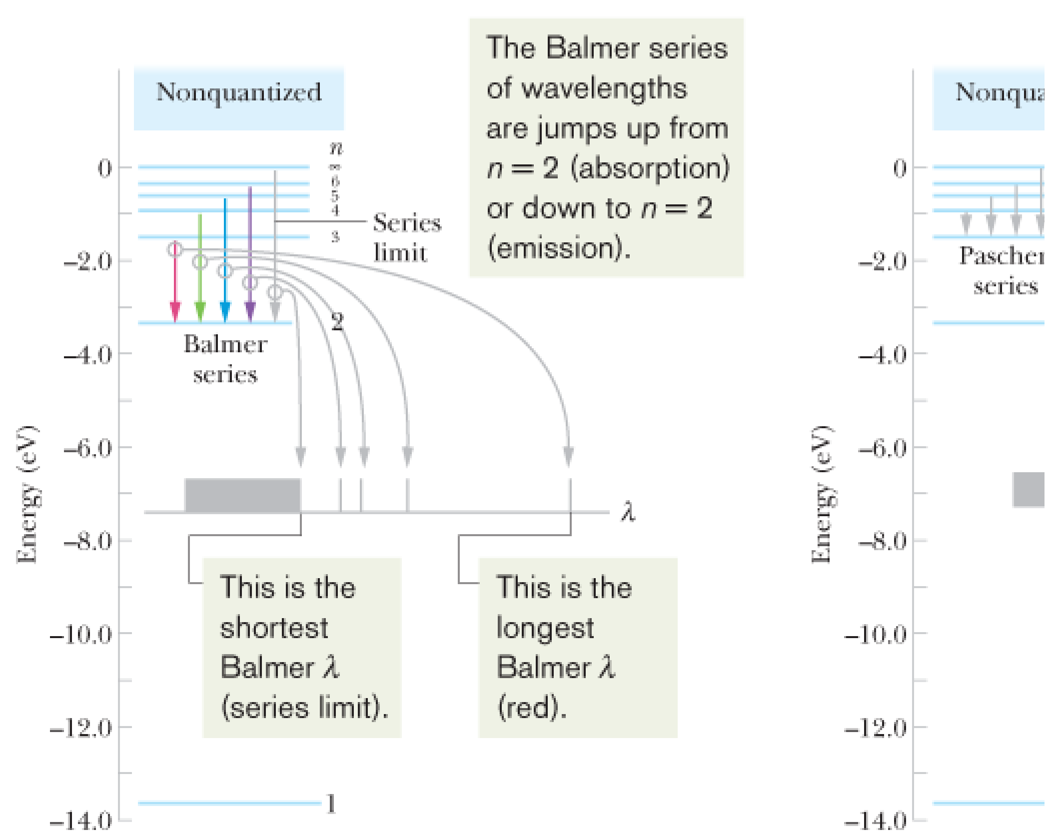
Figure 39-18*a* shows the energy levels corresponding to various values of n in Eq. 39-33. The lowest level, for $n = 1$, is the ground state of hydrogen. Higher levels correspond to excited states, just as we saw for our simpler potential traps. Note several differences, however. (1) The energy levels now have negative values rather than the positive values we previously chose in, for instance, Figs. 39-3 and 39-9. (2) The levels now become progressively closer as we move to higher levels. (3) The energy for the greatest value of n—namely, $n = \infty$—is now $E_\infty = 0$. For any energy greater than $E_\infty = 0$, the electron and proton are not bound together (there is no hydrogen atom), and the $E > 0$ region in Fig. 39-18*a* is like the nonquantized region for the finite well of Fig. 39-9.

Hydrogen emission series



Nonquantized
n
∞
6
5
4
3
2
1
0
–2.0
–4.0
–6.0
–8.0
–10.0
–12.0
–14.0
Energy (eV)
These are the lowest six allowed energies of the hydrogen atom.
Nonqua
Lyman se

(a)



Nonquantized
n
∞
6
5
4
3
2
1
Series limit
Balmer series
λ
0
–2.0
–4.0
–6.0
–8.0
–10.0
–12.0
–14.0
Energy (eV)
The Balmer series of wavelengths are jumps up from n = 2 (absorption) or down to n = 2 (emission).
This is the shortest Balmer λ (series limit).
This is the longest Balmer λ (red).
Nonqua
Paschen series

A hydrogen atom can jump between quantized energy levels by emitting or absorbing light at the wavelengths given by Eq. 39-36. Any such wavelength is often called a *line* because of the way it is detected with a spectroscope; thus, a hydrogen atom has *absorption lines* and *emission lines*. A collection of such lines, such as in those in the visible range, is called a **spectrum** of the hydrogen atom.

The lines for hydrogen are said to be grouped into *series*, according to the level at which upward jumps start and downward jumps end. For example, the emission and absorption lines for all possible jumps up from the $n = 1$ level and down to the $n = 1$ level are said to be in the *Lyman series* (Fig. 39-18*b*), named after the person who first studied those lines. Further, we can say that the Lyman series has a *home-base level* of $n = 1$. Similarly, the *Balmer series* has a home-base level of $n = 2$ (Fig. 39-18*c*), and the *Paschen series* has a home-base level of $n = 3$ (Fig. 39-18*d*).

Some of the downward quantum jumps for these three series are shown in Fig. 39-18. Four lines in the Balmer series are in the visible range and are the ones Balmer studied. They are represented in Fig. 39-18*c* with arrows corresponding to their colors. The shortest of those arrows represents the shortest jump in the series, from the $n = 3$ level to the $n = 2$ level. Thus, that jump involves the smallest change in the electron's energy and the smallest amount of emitted photon energy for the series. The emitted light is red. The next jump in the series, from $n = 4$ to $n = 2$, is longer, the photon energy is greater, the wavelength of the emitted light is shorter, and the light is green. The third, fourth, and fifth arrows represent longer jumps and shorter wavelengths. For the fifth jump, the emitted light is in the ultraviolet range and thus is not visible.

The *series limit* of a series is the line produced by the jump between the home-base level and the highest energy level, which is the level with quantum number $n = \infty$. Thus, the series limit is the shortest wavelength in the series.

If a jump is upward into the nonquantized portion of Fig. 39-18, the electron's energy is no longer given by Eq. 39-33 because the electron is no longer trapped in the atom. That is, the hydrogen atom has been *ionized*, meaning that the electron has been removed to a distance so great that the Coulomb force on it from the nucleus is negligible. The atom can be ionized if it absorbs any wavelength greater than the series limit. The free electron then has only kinetic energy K ($= \frac{1}{2}mv^2$, assuming a nonrelativistic situation).

Quantum Numbers for the Hydrogen Atom

Although the energies of the hydrogen atom states can be described by the single quantum number *n*, the wave functions describing these states require three quantum numbers, corresponding to the three dimensions in which the electron can move. The three quantum numbers, along with their names and the values that they may have, are shown in Table 39-2.

Table 39-2 Quantum Numbers for the Hydrogen Atom

Symbol	Name	Allowed Values
n	Principal quantum number	1,2,3, …
ℓ	Orbital quantum number	0,1,2,…, n - 1
m_ℓ	Orbital magnetic quantum number	$-\ell, -\ell(\ell - 1), \ldots +(\ell - 1), +\ell$

Each set of quantum numbers (n, ℓ, m_ℓ) identifies the wave function of a particular quantum state. The quantum number n, called the **principal quantum number**, appears in Eq. 39-33 for the energy of the state. The **orbital quantum number** ℓ is a measure of the magnitude of the angular momentum associated with the quantum state. The **orbital magnetic quantum number** m_ℓ is related to the orientation in space of this angular momentum vector. The restrictions on the values of the quantum numbers for the hydrogen atom, as listed in Table 39-2, are not arbitrary but come out of the solution to Schrödinger's equation. Note that for the ground state ($n = 1$), the restrictions require that $\ell = 0$ and $m_\ell = 0$. That is, the hydrogen atom in its ground state has zero angular momentum, which is not predicted by Eq. 39-24 in the Bohr model.

CHECKPOINT 5

(a) A group of quantum states of the hydrogen atom has $n = 5$. How many values of ℓ are possible for states in this group? (b) A subgroup of hydrogen atom states in the $n = 5$ group has $\ell = 3$. How many values of m_ℓ are possible for states in this subgroup?

Answer:

(a) 5;(b) 7

The Wave Function of the Hydrogen Atom's Ground State

The wave function for the ground state of the hydrogen atom, as obtained by solving the three-dimensional Schrödinger equation and normalizing the result, is

$$\psi(r) = \frac{1}{\sqrt{\pi}a^{3/2}} e^{-r/a} \quad \text{(ground state)}, \tag{39-39}$$

where a is the Bohr radius (Eq. 39-29). This radius is loosely taken to be the effective radius of a hydrogen atom and turns out to be a convenient unit of length for other situations involving atomic dimensions.

As with other wave functions, $\psi(r)$ in Eq. 39-39 does not have physical meaning but $\psi^2(r)$ does, being the probability density—the probability per unit volume—that the electron can be detected. Specifically, $\psi^2(r)\,dV$ is the probability that the electron can be detected in any given (infinitesimal) volume element dV located at radius r from the center of the atom:

$$\text{(probability of detection in volume } dV \text{ at radius } r) = \left(\text{volume probability density } \psi^2(r) \text{ a} \right. \tag{39-40}$$

Because $\psi^2(r)$ here depends only on r, it makes sense to choose, as a volume element dV, the volume between two concentric spherical shells whose radii are r and $r + dr$. That is, we take the volume element dV to be

$$dV = \left(4\pi r^2\right) dr, \tag{39-41}$$

in which $4\pi r^2$ is the surface area of the inner shell and dr is the radial distance between the two shells. Then, combining Eqs. 39-39, 39-40, and 39-41 gives us

$$(\text{probability of detection in volume } dV \text{ at radius } r) = \psi^2(r)dV = \frac{4}{a^3}e^{-2r/a}r^2dr. \quad (39\text{-}42)$$

Describing the probability of detecting an electron is easier if we work with a **radial probability density** $P(r)$ instead of a volume probability density $\psi^2(r)$. This $P(r)$ is a linear probability density such that

$$(\text{radial probability density } P(r) \text{ at radius } r)(\text{radial width } dr) = \left(\text{volume probability density } \psi^2\right.$$

or

$$P(r)dr = \psi^2(r)dV. \quad (39\text{-}43)$$

Substituting for $\psi^2(r)\,dV$ from Eq. 39-42, we obtain

$$P(r) = \frac{4}{a^3}r^2e^{-2r/a} \quad \text{(radial probability density, hydrogen atom ground state).} \quad (39\text{-}44)$$

Figure 39-19 is a plot of Eq. 39-44. The area under the plot is unity; that is,

$$\int_0^{\infty} P(r)\,dr = 1. \quad (39\text{-}45)$$

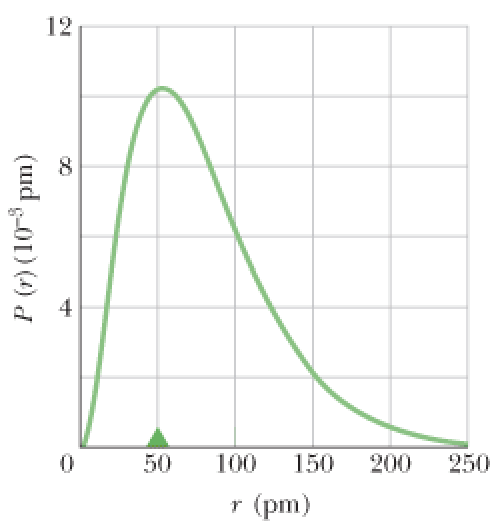


Figure 39-19 A plot of the radial probability density $P(r)$ for the ground state of the hydrogen atom. The triangular marker is located at one Bohr radius from the origin, and the origin represents the center of the atom.

This equation states that in a hydrogen atom, the electron must be *somewhere* in the space surrounding the nucleus.

The triangular marker on the horizontal axis of Fig. 39-19 is located one Bohr radius from the origin. The graph tells us that in the ground state of the hydrogen atom, the electron is most likely to be found at about this distance from the center of the atom.

Figure 39-19 conflicts sharply with the popular view that electrons in atoms follow well-defined orbits like planets moving around the Sun. *This popular view, however familiar, is incorrect* Figure 39-19 shows us all that we can ever know about the location of the electron in the ground state of the hydrogen atom. The appropriate question is not “When will the electron arrive at such-and-such a point?” but “What are the odds that the electron will be detected in a small volume centered on such-and-such a point?” Figure 39-20, which we call a dot plot, suggests the probabilistic nature of the wave function and provides a useful mental model of the hydrogen atom in its ground state. Think of the atom in this state as a fuzzy ball with no sharply defined boundary and no hint of orbits.

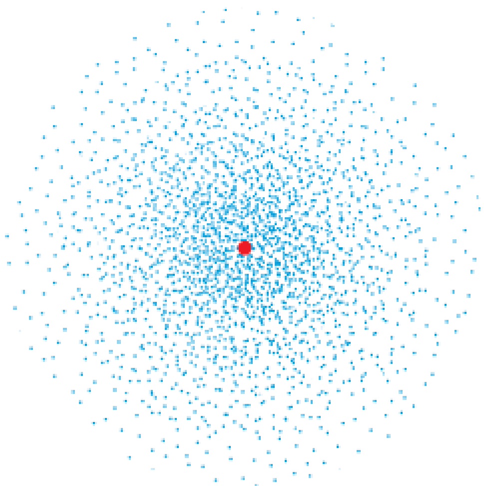


Figure 39-20A “dot plot” showing the volume probability density $\psi^2(r)$—not the *radial* probability density $P(r)$—for the ground state of the hydrogen atom. The density of dots drops exponentially with increasing distance from the nucleus, which is represented here by a red spot.

It is not easy for a beginner to envision subatomic particles in this probabilistic way. The difficulty is our natural impulse to regard an electron as something like a tiny jelly bean, located at certain places at certain times and following a well-defined path. Electrons and other subatomic particles simply do not behave in this way.

The energy of the ground state, found by putting $n = 1$ in Eq. 39-33, is $E_1 = -13.60$ eV. The wave function of Eq. 39-39 results if you solve Schrödinger's equation with this value of the energy. Actually, you can find a solution of Schrödinger's equation for *any* value of the energy—say, $E = -11.6$ eV or -14.3 eV. This may suggest that the energies of the hydrogen atom states are not quantized—but we know that they are.

The puzzle was solved when physicists realized that such solutions of Schrödinger's equation are not physically acceptable because they yield increasingly large values as $r \to \infty$. These “wave functions” tell us that the electron is more likely to be found very far from the nucleus rather than closer to it, which makes no sense. We get rid of these unwanted solutions by imposing what is called a **boundary condition**, in which we agree to accept only solutions of Schrödinger's equation for which $\psi(r) \to 0$ as $r \to \infty$; that is, we agree to deal only with *confined* electrons. With this restriction, the solutions of Schrödinger's equation form a discrete set, with quantized energies given by Eq. 39-33.

Radial probability density for the electron in a hydrogen atom

Show that the radial probability density for the ground state of the hydrogen atom has a maximum at *r* = *a*.

KEY IDEAS

(1) The radial probability density for a ground-state hydrogen atom is given by Eq. 39-44,

$$P(r) = \frac{4}{a^3} r^2 e^{-2r/a}.$$

(2) To find the maximum (or minimum) of any function, we must differentiate the function and set the result equal to zero.

Calculation:

If we differentiate *P*(*r*) with respect to *r*, using derivative 7 of Appendix E and the chain rule for differentiating products, we get

$$\begin{aligned} \frac{dP}{dr} &= \frac{4}{a^3} r^2 \left(\frac{-2}{a}\right) e^{-2r/a} + \frac{4}{a^3} 2re^{-2r/a} \\ &= \frac{8r}{a^3} e^{-2r/a} - \frac{8r^2}{a^4} e^{-2r/a} \\ &= \frac{8}{a^4} r(a-r) e^{-2r/a}. \end{aligned}$$

If we set the right side equal to zero, we obtain an equation that is true if *r* = *a*, so that the term (*a* = *r*) in the middle of the equation is zero. In other words, *dP/dr* is equal to zero when *r* = *a*. (Note that we also have *dP/dr* = 0 at *r* = 0 and at *r* = ∞. However, these conditions correspond to a minimum in *P*(*r*), as you can see in Fig. 39-19.)

Probability of detection of the electron in a hydrogen atom

It can be shown that the probability *p*(*f*) that the electron in the ground state of the hydrogen atom will be detected inside a sphere of radius *r* is given by

$$p(r) = 1 - e^{-2x}\left(1 + 2x + 2x^2\right),$$

in which *x*, a dimensionless quantity, is equal to *r*/*a*. Find *r* for *p*(*r*) = 0.90.

KEY IDEA

There is no guarantee of detecting the electron at any particular radial distance *r* from the center of the hydrogen atom. However, with the given function, we can calculate the probability that the electron will be detected *somewhere* within a sphere of radius *r*.

Calculation:

We seek the radius of a sphere for which $p(r) = 0.90$. Substituting that value in the expression for $p(r)$, we have

$$0.90 = 1 - e^{-2x}\left(1 + 2x + 2x^2\right)$$

or

$$10e^{-2x}\left(1 + 2x + 2x^2\right) = 1.$$

We must find the value of *x* that satisfies this equality. It is not possible to solve explicitly for *x*, but an equation solver on a calculator yields $x = 2.66$. This means that the radius of a sphere within which the electron will be detected 90% of the time is 2.66*a*. Mark this position on the horizontal axis of Fig. 39-19. The area under the curve from $r = 0$ to $r = 2.66a$ gives the probability of detection in that range and is 90% of the total area under the curve.

Light emission from a hydrogen atom

(a) What is the wavelength of light for the least energetic photon emitted in the Lyman series of the hydrogen atom spectrum lines?

KEY IDEAS

(1) For any series, the transition that produces the least energetic photon is the transition between the home-base level that defines the series and the level immediately above it. (2) For the Lyman series, the home-base level is at $n = 1$ (Fig. 39-18*b*). Thus, the transition that produces the least energetic photon is the transition from the $n = 2$ level to the $n = 1$ level.

Calculations:

From Eq. 39-33 the energy difference is

$$\Delta E = E_2 - E_1 = -(13.60\text{eV})\left(\frac{1}{2^2} - \frac{1}{1^2}\right) = 10.20\text{eV}.$$

Then from Eq. 39-6 ($\Delta E = hf$), with c/λ replacing f, we have

$$\lambda = \frac{hc}{\Delta E} = \frac{(6.63 \times 10^{-34}\,\mathrm{J \cdot s})(300 \times 10^{8}\,\mathrm{m/s})}{(10.20\,\mathrm{eV})(1.60 \times 10^{-19}\,\mathrm{J/eV})}$$

$$= 1.22 \times 10^{-7}\,\mathrm{m} = 122\,\mathrm{nm}.$$ (Answer)

Light with this wavelength is in the ultraviolet range.

(b) What is the wavelength of the series limit for the Lyman series?

KEY IDEA

The series limit corresponds to a jump between the home-base level (n = 1 for the Lyman series) and the level at the limit $n = \infty$.

Calculations:

Now that we have identified the values of n for the transition, we could proceed as in (a) to find the corresponding wavelength λ. Instead, let's use a more direct procedure. From Eq. 39-36, we find

$$\frac{1}{\lambda} = R\left(\frac{1}{n_{\text{low}}^2} - \frac{1}{n_{\text{high}}^2}\right)$$

$$= 1.097\,373 \times 10^{7}\,\mathrm{m}^{-1}\left(\frac{1}{1^2} - \frac{1}{\infty^2}\right),$$

which yields

$$\lambda = 9.11 \times 10^{-8}\,\mathrm{m} = 91.1\,\mathrm{nm}.$$ (Answer)

Light with this wavelength is also in the ultraviolet range.

Hydrogen Atom States with *n* = 2

According to the requirements of Table 39-2, there are four states of the hydrogen atom with n = 2; their quantum numbers are listed in Table 39-3. Consider first the state with $n = 2$ and $\ell = m_\ell = 0$; its probability density is represented by the dot plot of Fig. 39-21. Note that this plot, like the plot for the ground state shown in Fig. 39-20, is spherically symmetric. That is, in a spherical coordinate system like that defined in Fig. 39-22, the probability density is a function of the radial coordinate r only and is independent of the angular coordinates θ and .

Table 39-3 Quantum Numbers for Hydrogen Atom States with *n* = 2

n	ℓ	m_ℓ
2	0	0

2	1	+1
2	1	0
2	1	-1

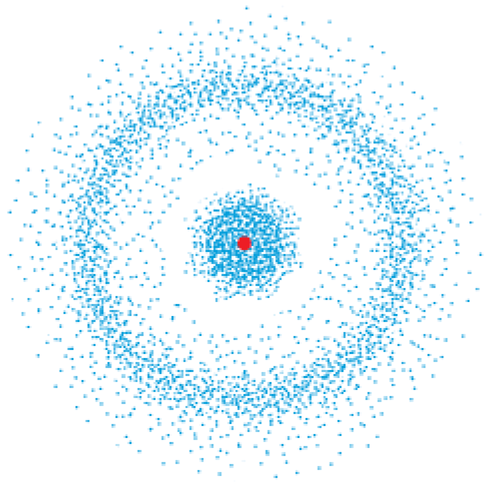


Figure 39-21 A dot plot showing the volume probability density $\psi^2(r)$ for the hydrogen atom in the quantum state with $n = 2$, $\ell = 0$, and $m_\ell = 0$. The plot has spherical symmetry about the central nucleus. The gap in the dot density pattern marks a spherical surface over which $\psi^2(r) = 0$.

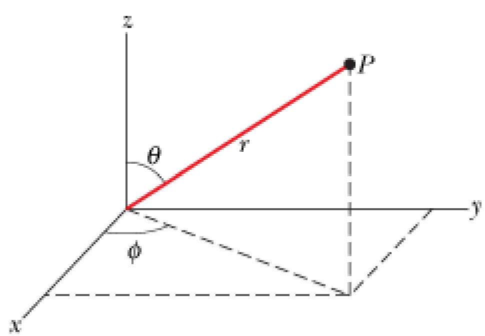


Figure 39-22 The relationship between the coordinates x, y, and z of the rectangular coordinate system and the coordinates r, θ, and of the spherical coordinate system. The latter are more appropriate for analyzing situations involving spherical symmetry, such as the hydrogen atom.

It turns out that all quantum states with $\ell = 0$ have spherically symmetric wave functions. This is reasonable because the quantum number ℓ is a measure of the angular momentum associated with a given state. If $\ell = 0$, the angular momentum is also zero, which requires that the probability density representing the state have no preferred axis of symmetry.

Dot plots of ψ^2 for the three states with $n = 2$ and $\ell = 1$ are shown in Fig. 39-23. The probability densities for the states with $m_\ell = +1$ and $m_\ell = -1$ are identical. Although these plots are symmetric about the z axis, they are *not* spherically symmetric. That is, the probability densities for these three states are functions of both r and the angular coordinate θ.

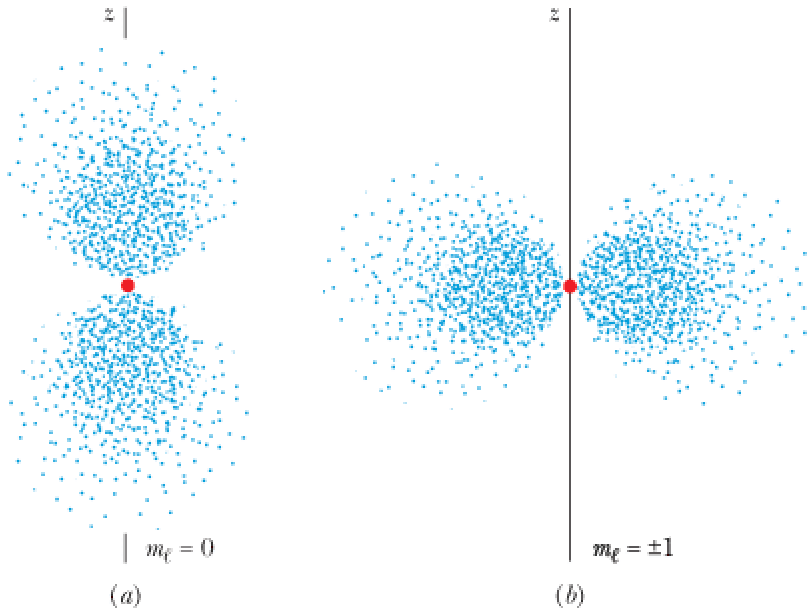


Figure 39-23 Dot plots of the volume probability density $\psi^2(r, \theta)$ for the hydrogen atom in states with $n = 2$ and $\ell = 1$. (a) Plot for $m_\ell = 0$. (b) Plot for $m_\ell = +1$ and $m_\ell = -1$. Both plots show that the probability density is symmetric about the z axis.

Here is a puzzle: What is there about the hydrogen atom that establishes the axis of symmetry that is so obvious in Fig. 39-23? The answer: *absolutely nothing.*

The solution to this puzzle comes about when we realize that all three states shown in Fig. 39-23 have the same energy. Recall that the energy of a state, given by Eq. 39-33, depends only on the principal quantum number n and is independent of ℓ and m_ℓ. In fact, for an *isolated* hydrogen atom there is no way to differentiate experimentally among the three states of Fig. 39-23.

If we add the volume probability densities for the three states for which $n = 2$ and $\ell = 1$, the combined probability density turns out to be spherically symmetrical, with no unique axis. One can, then, think of the electron as spending one-third of its time in each of the three states of Fig. 39-23, and one can think of the weighted sum of the three independent wave functions as defining a spherically symmetric **subshell** specified by the quantum numbers $n = 2$, $\ell = 1$. The individual states will display their separate existence only if we place the hydrogen atom in an external electric or magnetic field. The three states of the $n = 2$, $\ell = 1$ subshell will then have different energies, and the field direction will establish the necessary symmetry axis.

The $n = 2$, $\ell = 0$ state, whose volume probability density is shown in Fig. 39-21, *also* has the same energy as each of the three states of Fig. 39-23. We can view all four states whose quantum numbers are listed in Table 39-3 as forming a spherically symmetric **shell** specified by the single quantum number n. The importance of shells and subshells will become evident in Chapter 40, where we discuss atoms having more than one electron.

To round out our picture of the hydrogen atom, we display in Fig. 39-24 a dot plot of the *radial* probability density for a hydrogen atom state with a relatively high quantum number ($n = 45$) and the highest orbital quantum number that the restrictions of Table 39-2 permit ($\ell = n - 1 = 44$). The probability density forms a ring that is symmetrical about the z axis and lies very close to the xy plane. The mean radius of the ring is n^2a, where a is the Bohr radius. This mean radius is more than 2000 times the effective radius of the hydrogen atom in its ground state.

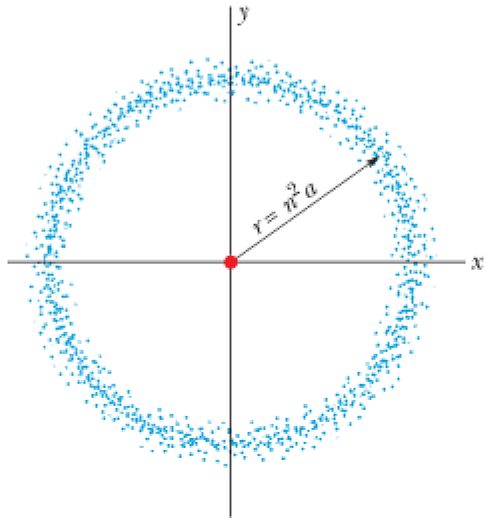


Figure 39-24 A dot plot of the radial probability density $P(r)$ for the hydrogen atom in a quantum state with a relatively large principal quantum number—namely, $n = 45$—and angular momentum quantum number $\ell = n - 1 = 44$. The dots lie close to the *xy* plane, the ring of dots suggesting a classical electron orbit.

Figure 39-24 suggests the electron orbit of classical physics - it resembles the circular orbit of a planet around a star. Thus, we have another illustration of Bohr's correspondence principle - namely, that at large quantum numbers the predictions of quantum mechanics merge smoothly with those of classical physics. Imagine what a dot plot like that of Figure 39-24 would look like for *really* large values of n and ℓ - say, $n = 1000$ and $\ell = 999$.

sec. 39-3 Energies of a Trapped Electron

•1 An electron in a one-dimensional infinite potential well of length L has ground-state energy E_1. The length is changed to L' so that the new ground-state energy is $E'_1 = 0.500E_1$. What is the ratio L'/L?

Answer:

1.41

•2 What is the ground-state energy of (a) an electron and (b) a proton if each is trapped in a one-dimensional infinite potential well that is 200 pm wide?

•3 The ground-state energy of an electron trapped in a one-dimensional infinite potential well is 2.6 eV. What will this quantity be if the width of the potential well is doubled?

Answer:

0.65 eV

•4 An electron, trapped in a one-dimensional infinite potential well 250 pm wide, is in its ground state. How much energy must it absorb if it is to jump up to the state with $n = 4$?

•5 What must be the width of a one-dimensional infinite potential well if an electron trapped in it in the $n = 3$ state is to have an energy of 4.7 eV?

Answer:

0.85 nm

•6 A proton is confined to a one-dimensional infinite potential well 100 pm wide. What is its ground-state energy?

•7 Consider an atomic nucleus to be equivalent to a one-dimensional infinite potential well with $L = 1.4 \times 10^{-14}$ m, a typical nuclear diameter. What would be the ground-state energy of an electron if it were trapped in such a potential well? (*Note:* Nuclei do not contain electrons.)

Answer:

1.9 GeV

••8 GO An electron is trapped in a one-dimensional infinite well and is in its first excited state. Figure 39-27 indicates the five longest wavelengths of light that the electron could absorb in transitions from this initial state via a single photon absorption: $\lambda_a = 80.78$ nm, $\lambda_b = 33.66$ nm, $\lambda_c = 19.23$ nm, $\lambda_d = 12.62$ nm, and $\lambda_e = 8.98$ nm. What is the width of the potential well?

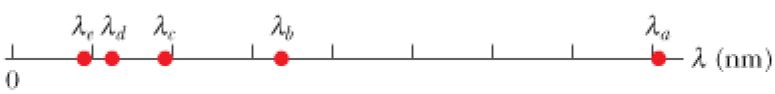


Figure 39-27 Problem 8.

••9 Suppose that an electron trapped in a one-dimensional infinite well of width 250 pm is excited from its first excited state to its third excited state. (a) What energy must be transferred to the electron for this quantum jump? The electron then de-excites back to its ground state by emitting light. In the various possible ways it can do this, what are the (b) shortest, (c) second shortest, (d) longest, and (e) second longest wavelengths that can be emitted? (f) Show the various possible ways on an energy-level diagram. If light of wavelength 29.4 nm happens to be emitted, what are the (g) longest and (h) shortest wavelength that can be emitted afterwards?

Answer:

(a) 72.2 eV; (b) 13.7 nm; (c) 17.2 nm; (d) 68.7 nm; (e) 41.2 nm; (g) 68.7 nm; (h) 25.8 nm

••10 An electron is trapped in a one-dimensional infinite potential well. For what (a) higher quantum number and (b) lower quantum number is the corresponding energy difference equal to the energy difference ΔE_{43} between the levels $n = 4$ and $n = 3$? (c) Show that no pair of adjacent levels has an energy difference equal to $2\Delta E_{43}$.

••11 An electron is trapped in a one-dimensional infinite potential well. For what (a) higher quantum number and (b) lower quantum number is the corresponding energy difference equal to the energy of the $n = 5$ level? (c) Show that no pair of adjacent levels has an energy difference equal to the energy of the $n = 6$ level.

Answer:

(a) 13; (b) 12

••12 GO An electron is trapped in a one-dimensional infinite well of width 250 pm and is in its ground state. What are the (a) longest, (b) second longest, and (c) third longest wavelengths of light that can excite the electron from the ground state via a single photon absorption?

sec. 39-4 Wave Functions of a Trapped Electron

••13 GO A one-dimensional infinite well of length 200 pm contains an electron in its third excited state. We position an electron-detector probe of width 2.00 pm so that it is centered on a point of maximum probability density. (a) What is the probability of detection by the probe? (b) If we insert the probe as described 1000 times, how many times should we expect the electron to materialize on the end of the probe (and thus be detected)?

Answer:

(a) 0.020; (b) 20

••14 An electron is in a certain energy state in a one-dimensional, infinite potential well from $x = 0$ to $x = L = 200$ pm. The electron's probability density is zero at $x = 0.300L$, and $x = 0.400L$; it is not zero at intermediate values of x. The electron then jumps to the next lower energy level by emitting light. What is the change in the electron's energy?

••15 SSM WWW An electron is trapped in a one-dimensional infinite potential well that is 100 pm wide; the electron is in its ground state. What is the probability that you can detect the electron in an interval of width $\Delta x = 5.0$ pm centered at $x =$ (a) 25 pm, (b) 50 pm, and (c) 90 pm? (*Hint:* The interval Δx is so narrow that you can take the probability density to be constant within it.)

Answer:

(a) 0.050; (b) 0.10; (c) 0.0095

••16 A particle is confined to the one-dimensional infinite potential well of Fig. 39-2. If the particle is in its ground state, what is its probability of detection between (a) $x = 0$ and $x = 0.25L$, (b) $x = 0.75L$ and $x = L$, and (c) $x = 0.25L$ and $x = 0.75L$?

sec. 39-5 An Electron in a Finite Well

•17 An electron in the $n = 2$ state in the finite potential well of Fig. 39-7 absorbs 400 eV of energy from an external source. Using the energy-level diagram of Fig. 39-9, determine the electron's kinetic energy after this absorption, assuming that the electron moves to a position for which $x > L$.

Answer:

56 eV

•18 Figure 39-9 gives the energy levels for an electron trapped in a finite potential energy well 450 eV deep. If the electron is in the $n = 3$ state, what is its kinetic energy?

••19 Figure 39-28*a* shows the energy-level diagram for a finite, one-dimensional energy well that contains an electron. The non-quantized region begins at $E_4 = 450.0$ eV. Figure 39-28*b* gives the absorption spectrum of the electron when it is in the ground state—it can absorb at the indicated wavelengths: $\lambda_a = 14.588$ nm and $\lambda_b = 4.8437$ nm and for any wavelength less than $\lambda_c = 2.9108$

nm. What is the energy of the first excited state?

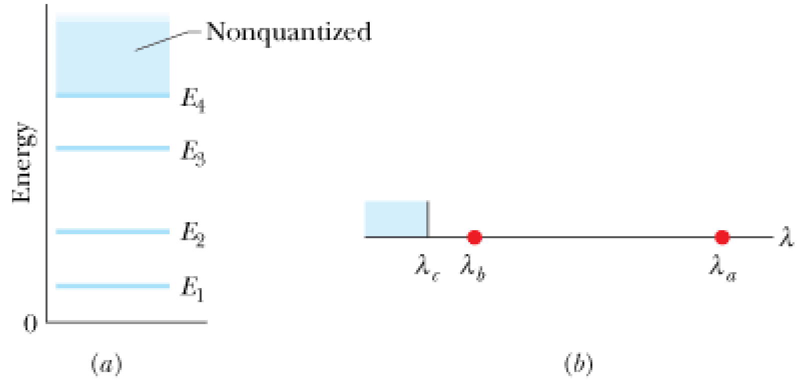


Figure 39-28 Problem 19.

Answer:

109 eV

••20Figure 39-29*a* shows a thin tube in which a finite potential trap has been set up where $V_2 = 0$ V. An electron is shown traveling rightward toward the trap, in a region with a voltage of $V_1 = -9.00$ V, where it has a kinetic energy of 2.00 eV. When the electron enters the trap region, it can become trapped if it gets rid of enough energy by emitting a photon. The energy levels of the electron within the trap are $E_1 = 1.0$, $E_2 = 2.0$, and $E_3 = 4.0$ eV, and the nonquantized region begins at $E_4 = 9.0$ eV as shown in the energy-level diagram of Fig. 39-29*b*. What is the smallest energy (eV) such a photon can have?

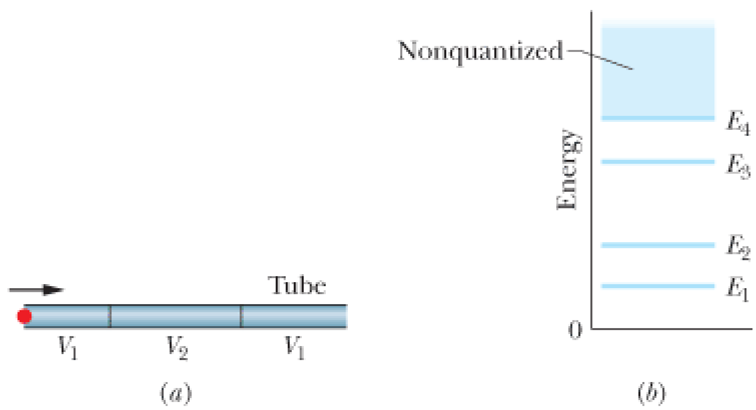


Figure 39-29 Problem 20.

••21(a) Show that for the region $x > L$ in the finite potential well of Fig. 39-7, $\psi(x) = De^{2kx}$ is a solution of Schrödinger's equation in its one-dimensional form, where D is a constant and k is positive. (b) On what basis do we find this mathematically acceptable solution to be physically unacceptable?

sec. 39-7 Two- and Three-Dimensional Electron Traps

•22 GO An electron is contained in the rectangular corral of Fig. 39-13, with widths $L_x = 800$ pm and $L_y = 1600$ pm. What is the electron's ground-state energy?

•23An electron is contained in the rectangular box of Fig. 39-14, with widths $L_x = 800$ pm, $L_y = 1600$ pm, and $L_z = 390$ pm. What is the electron's ground-state energy?

Answer:

3.21 eV

••24 Figure 39-30 shows a two-dimensional, infinite-potential well lying in an xy plane that contains an electron. We probe for the electron along a line that bisects L_x and find three points at which the detection probability is maximum. Those points are separated by 2.00 nm. Then we probe along a line that bisects L_y and find five points at which the detection probability is maximum. Those points are separated by 3.00 nm. What is the energy of the electron?

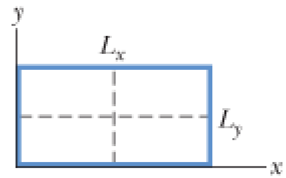


Figure 39-30 Problem 24.

••25 GO The two-dimensional, infinite corral of Fig. 39-31 is square, with edge length L = 150 pm. A square probe is centered at xy coordinates (0.200L, 0.800L) and has an x width of 5.00 pm and a y width of 5.00 pm. What is the probability of detection if the electron is in the $E_{1,3}$ energy state?

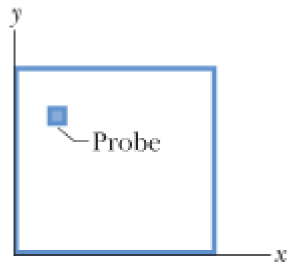


Figure 39-31 Problem 25.

Answer:

1.4×10^{-3}

••26 A rectangular corral of widths $L_x = L$ and $L_y = 2L$ contains an electron. What multiple of $h^2/8mL^2$, where m is the electron mass, gives (a) the energy of the electron's ground state, (b) the energy of its first excited state, (c) the energy of its lowest degenerate states, and (d) the difference between the energies of its second and third excited states?

••27 SSM WWW An electron (mass m) is contained in a rectangular corral of widths $L_x = L$ and $L_y = 2L$. (a) How many different frequencies of light could the electron emit or absorb if it makes a transition between a pair of the lowest five energy levels? What multiple of $h/8mL^2$ gives the (b) lowest, (c) second lowest, (d) third lowest, (e) highest, (f) second highest, and (g) third highest frequency?

Answer:

(a) 8; (b) 0.75; (c) 1.00; (d) 1.25; (e) 3.75; (f) 3.00; (g) 2.25

••28 GO A cubical box of widths $L_x = L_y = L_z = L$ contains an electron. What multiple of $h^2/8mL^2$, where m is the electron mass, is (a) the energy of the electron's ground state, (b) the energy of its second excited state, and (c) the difference between the energies of its second and third excited states? How many degenerate states have the energy of (d) the first excited state and (e) the fifth excited state?

••29 An electron (mass m) is contained in a cubical box of widths $L_x = L_y = L_z$. (a) How many different frequencies of light could the electron emit or absorb if it makes a transition between a pair of the lowest five energy levels? What multiple of $h/8mL^2$ gives the (b) lowest, (c) second lowest, (d) third lowest, (e) highest, (f) second highest, and (g) third highest frequency?

Answer:

(a) 7; (b) 1.00; (c) 2.00; (d) 3.00; (e) 9.00; (f) 8.00; (g) 6.00

•••30 An electron is in the ground state in a two-dimensional, square, infinite potential well with edge lengths L. We will probe for it in a square of area 400 pm^2 that is centered at $x = L/8$ and $y = L/8$. The probability of detection turns out to be 4.5×10^{-8}. What is edge length L?

sec. 39-9 Schrödinger's Equation and the Hydrogen Atom

•31 SSM What is the ratio of the shortest wavelength of the Balmer series to the shortest wavelength of the Lyman series?

Answer:

4.0

•32 An atom (not a hydrogen atom) absorbs a photon whose associated wavelength is 375 nm and then immediately emits a photon whose associated wavelength is 580 nm. How much net energy is absorbed by the atom in this process?

•33 What are the (a) energy, (b) magnitude of the momentum, and (c) wavelength of the photon emitted when a hydrogen atom undergoes a transition from a state with $n = 3$ to a state with $n = 1$?

Answer:

(a) 12.1 eV;(b) 6.45×10^{-27} kg · m/s;(c) 102 nm

•34 Calculate the radial probability density $P(r)$ for the hydrogen atom in its ground state at (a) $r = 0$, (b) $r = a$, and (c) $r = 2a$, where a is the Bohr radius.

•35 For the hydrogen atom in its ground state, calculate (a) the probability density $\psi^2(r)$ and (b) the radial probability density $P(r)$ for $r = a$, where a is the Bohr radius.

Answer:

(a) 291 nm^{-3}; (b) 10.2 nm^{-1}

•36 (a) What is the energy E of the hydrogen-atom electron whose probability density is represented by the dot plot of Fig. 39-21? (b) What minimum energy is needed to remove this electron from the atom?

•37 SSM A neutron with a kinetic energy of 6.0 eV collides with a stationary hydrogen atom in its ground state. Explain why the collision must be elastic—that is, why kinetic energy must be conserved. (*Hint:* Show that the hydrogen atom cannot be excited as a result of the collision.)

•38 An atom (not a hydrogen atom) absorbs a photon whose associated frequency is 6.2×10^{14} Hz. By what amount does the energy of the atom increase?

••39 SSM Verify that Eq. 39-44, the radial probability density for the ground state of the hydrogen atom, is normalized. That is, verify that the following is true:

$$\int_0^\infty P(r)\,dr = 1 \qquad (39\text{-}46)$$

••40 What are the (a) wavelength range and (b) frequency range of the Lyman series? What are the (c) wavelength range and (d) frequency range of the Balmer series?

••41 What is the probability that an electron in the ground state of the hydrogen atom will be found between two spherical shells whose radii are r and $r + \Delta r$, (a) if $r = 0.500a$ and $\Delta r = 0.010a$ and (b) if $r = 1.00a$ and $\Delta r = 0.01a$, where a is the Bohr radius? (*Hint:* Δr is small enough to permit the radial probability density to be taken to be constant between r and $r + \Delta r$.)

Answer:

(a) 0.0037; (b) 0.0054

••42 A hydrogen atom, initially at rest in the $n = 4$ quantum state, undergoes a transition to the ground state, emitting a photon in the process. What is the speed of the recoiling hydrogen atom? (*Hint:* This is similar to the explosions of Chapter 9.)

••43 In the ground state of the hydrogen atom, the electron has a total energy of -13.6 eV. What are (a) its kinetic energy and (b) its potential energy if the electron is one Bohr radius from the central nucleus?

Answer:

(a) 13.6 eV; (b) -27.2 eV

••44 A hydrogen atom in a state having a *binding energy* (the energy required to remove an electron) of 0.85 eV makes a transition to a state with an *excitation energy* (the difference between the energy of the state and that of the ground state) of 10.2 eV. (a) What is the energy of the photon emitted as a result of the transition? What are the (b) higher quantum number and (c) lower quantum number of the transition producing this emission?

••45 SSM The wave functions for the three states with the dot plots shown in Fig. 39-23, which have $n = 2$, $\ell = 1$, and $m_\ell = 0, +1$, and -1, are

$$\begin{aligned}
\psi_{210}(r,\theta) &= \left(1/4\sqrt{2\pi}\right)\left(a^{-3/2}\right)(r/a)e^{-r/2a}\cos\theta,\\
\psi_{21+1}(r,\theta) &= \left(1/8\sqrt{\pi}\right)\left(a^{-3/2}\right)(r/a)e^{-r/2a}(\sin\theta)e^{+i\phi},\\
\psi_{21-1}(r,\theta) &= \left(1/8\sqrt{\pi}\right)\left(a^{-3/2}\right)(r/a)e^{-r/2a}(\sin\theta)e^{-i\phi},
\end{aligned}$$

in which the subscripts on $\psi(r, 0)$ give the values of the quantum numbers n, ℓ, m_ℓ and the angles θ and are defined in Fig. 39-22. Note that the first wave function is real but the others, which involve the imaginary number i, are complex. Find the radial probability density $P(r)$ for (a) ψ_{210} and (b) ψ_{21+1} (same as for ψ_{21-1}). (c) Show that each $P(r)$ is consistent with the corresponding dot plot in Fig. 39-23. (d) Add the radial probability densities for ψ_{210}, ψ_{21+1}, and ψ_{21-1} and then show that the sum is spherically symmetric, depending only on r.

Answer:

(a) $(r^4/8a^5)[\exp(-r/a)]\cos^2\theta$; (b) $(r^4/16a^5)[\exp(-r/a)]\sin^2\theta$

••46 Calculate the probability that the electron in the hydrogen atom, in its ground state, will be found between spherical shells whose radii are a and $2a$, where a is the Bohr radius.

••47 For what value of the principal quantum number n would the effective radius, as shown in a probability density dot plot for the hydrogen atom, be 1.0 mm? Assume that i has its maximum value of n - 1. (*Hint:* See Fig. 39-24.)

Answer:

4.3×10^3

••48 Light of wavelength 121.6 nm is emitted by a hydrogen atom. What are the (a) higher quantum number and (b) lower quantum number of the transition producing this emission? (c) What is the name of the series that includes the transition?

••49 How much work must be done to pull apart the electron and the proton that make up the hydrogen atom if the atom is initially in (a) its ground state and (b) the state with $n = 2$?

Answer:

(a) 13.6 eV; (b) 3.40 eV

••50 Light of wavelength 102.6 nm is emitted by a hydrogen atom. What are the (a) higher quantum number and (b) lower quantum number of the transition producing this emission? (c) What is the name of the series that includes the transition?

••51 What is the probability that in the ground state of the hydrogen atom, the electron will be found at a radius greater than the Bohr radius?)

Answer:

0.68

••52 A hydrogen atom is excited from its ground state to the state with $n = 4$. (a) How much energy must be absorbed by the atom? Consider the photon energies that can be emitted by the atom as it de-excites to the ground state in the several possible ways. (b) How many different energies are possible; what are the (c) highest, (d) second highest, (e) third highest, (f) lowest, (g) second lowest, and (h) third lowest energies?

••53 SSM WWW Schrödinger's equation for states of the hydrogen atom for which the orbital quantum number ℓ is zero is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right)+\frac{8\pi^2 m}{h^2}[E-U(r)]\psi=0.$$

Verify that Eq. 39-39, which describes the ground state of the hydrogen atom, is a solution of this equation.

•••54 The wave function for the hydrogen-atom quantum state represented by the dot plot shown in Fig. 39-21, which has $n = 2$ and $\ell = m_\ell = 0$, is

$$\psi_{200}(r)=\frac{4}{4\sqrt{2\pi}}a^{-3/2}\left(2-\frac{r}{a}\right)e^{-r/2a},$$

in which a is the Bohr radius and the subscript on $\psi(r)$ gives the values of the quantum numbers n, ℓ, m_ℓ. (a) Plot $\psi^2_{200}(r)$ and show that your plot is consistent with the dot plot of Fig. 39-21. (b) Show analytically that $\psi^2_{200}(r)$ has a maximum at $r = 4a$. (c) Find the radial probability density

$P_{200}(r)$ for this state. (d) Show that

$$\int_0^\infty P_{200}(r)\,dr = 1$$

and thus that the expression above for the wave function $\psi_{200}(r)$ has been properly normalized.

•••55The radial probability density for the ground state of the hydrogen atom is a maximum when $r = a$, where a is the Bohr radius. Show that the *average* value of r, defined as

$$r_{avg} = \int P(r)r\,dr,$$

has the value $1.5a$. In this expression for r_{avg}, each value of $P(r)$ is weighted with the value of r at which it occurs. Note that the average value of r is greater than the value of r for which $P(r)$ is a maximum.

Additional Problems

56Let ΔE_{adj} be the energy difference between two adjacent energy levels for an electron trapped in a one-dimensional infinite potential well. Let E be the energy of either of the two levels. (a) Show that the ratio $\Delta E_{adj}/E$ approaches the value $2/n$ at large values of the quantum number n. As $n \rightarrow \infty$, does (b) ΔE_{adj}, (c) E, or (d) $\Delta E_{adj}/E$ approach zero? (e) What do these results mean in terms of the correspondence principle?

57An electron is trapped in a one-dimensional infinite potential well. Show that the energy difference *A.E* between its quantum levels n and $n + 2$ is $(h^2/2mL^2)(n + 1)$.

58As Fig. 39-8 suggests, the probability density for an electron in the region $0 < x < L$ for the finite potential well of Fig. 39-7 is sinusoidal, being given by $\psi^2(x) = B \sin^2 kx$, in which B is a constant. (a) Show that the wave function *i(i(x)* that may be found from this equation is a solution of Schrödinger's equation in its one-dimensional form. (b) Find an expression for k that makes this true.

59 SSM As Fig. 39-8 suggests, the probability density for the region $x > L$ in the finite potential well of Fig. 39-7 drops off exponentially according to $\psi^2(x) = Ce^{-2kx}$, where C is a constant. (a) Show that the wave function *i(i(x)* that may be found from this equation is a solution of Schrödinger's equation in its one-dimensional form. (b) Find an expression for k for this to be true.

Answer:

(b) $(2\pi/h)[2m(U_0 - E)]^{0.5}$

60An electron is confined to a narrow evacuated tube of length 3.0 m; the tube functions as a one-dimensional infinite potential well. (a) What is the energy difference between the electron's ground state and its first excited state? (b) At what quantum number n would the energy difference between adjacent energy levels be 1.0 eV—which is measurable, unlike the result of (a)? At that quantum number, (c) what multiple of the electron's rest energy would give the electron's total energy and (d) would the electron be relativistic?

61(a) Show that the terms in Schrödinger's equation (Eq. 39-18) have the same dimensions. (b) What is the common SI unit for each of these terms?

Answer:

(b) meter$^{-2.5}$

62(a) What is the wavelength of light for the least energetic photon emitted in the Balmer series of the hydrogen atom spectrum lines? (b) What is the wavelength of the series limit?

63(a) For a given value of the principal quantum number n for a hydrogen atom, how many values of the orbital quantum number ℓ are possible? (b) For a given value of ℓ, how many values of the orbital magnetic quantum number m_ℓ are possible? (c) For a given value of n, how many values of m_ℓ are possible?

Answer:

(a) n; (b) $2\ell + 1$; (c) n^2

64 Verify that the combined value of the constants appearing in Eq. 39-32 is 13.6 eV

CHAPTER 44 QUARKS, LEPTONS, AND THE BIG BANG

44-1 What is Physics?

Physicists often refer to the theories of relativity and quantum physics as “modern physics,” to distinguish them from the theories of Newtonian mechanics and Maxwellian electromagnetism, which are lumped together as “classical physics.” As the years go by, the word “modern” seems less and less appropriate for theories whose foundations were laid down in the opening years of the 20th century. After all, Einstein published his paper on the photoelectric effect and his first paper on special relativity in 1905, Bohr published his quantum model of the hydrogen atom in 1913, and Schrödinger published his matter wave equation in 1926. Nevertheless, the label of “modern physics” hangs on.

In this closing chapter we consider two lines of investigation that are truly “modern” but at the same time have the most ancient of roots. They center around two deceptively simple questions:

What is the universe made of?

How did the universe come to be the way it is?

Progress in answering these questions has been rapid in the last few decades.

Many new insights are based on experiments carried out with large particle accelerators. However, as they bang particles together at higher and higher energies using larger and larger accelerators, physicists come to realize that no conceivable Earth-bound accelerator can generate particles with energies great enough to test the ultimate theories of physics. There has been only one source of particles with these energies, and that was the universe itself within the first millisecond of its existence.

In this chapter you will encounter a host of new terms and a veritable flood of particles with names that you should not try to remember. If you are temporarily bewildered, you are sharing the bewilderment of the physicists who lived through these developments and who at times saw nothing but increasing complexity with little hope of understanding. If you stick with it, however, you will come to share the excitement physicists felt as marvelous new accelerators poured out new results, as the theorists put forth ideas each more daring than the last, and as clarity finally sprang from obscurity. The main message of this book is that, although we know a lot about the physics of the world, grand mysteries remain.

44-2 Particles, Particles, Particles

In the 1930s, there were many scientists who thought that the problem of the ultimate structure of matter was well on the way to being solved. The atom could be understood in terms of only three particles—the electron, the proton, and the neutron. Quantum physics accounted well for the structure of the atom and for radioactive alpha decay. The neutrino had been postulated and, although not yet observed, had been incorporated by Enrico Fermi into a successful theory of beta decay. There was hope that quantum theory applied to protons and neutrons would soon account for the structure of the nucleus. What else was there?

The euphoria did not last. The end of that same decade saw the beginning of a period of discovery of new particles that continues to this day. The new particles have names and symbols such as *muon* (μ), *pion* (π), *kaon* (K), and *sigma* (Σ). All the new particles are unstable; that is, they spontaneously transform into other types of particles according to the same functions of time that apply to unstable nuclei. Thus, if N_0 particles of any one type are present in a sample at time $t = 0$, the number N of those particles present at some later time t is given by Eq. 42-15,

$$N = N_0 e^{-\lambda t}, \tag{44-1}$$

the rate of decay R, from an initial value of R_0, is given by Eq. 42-16,

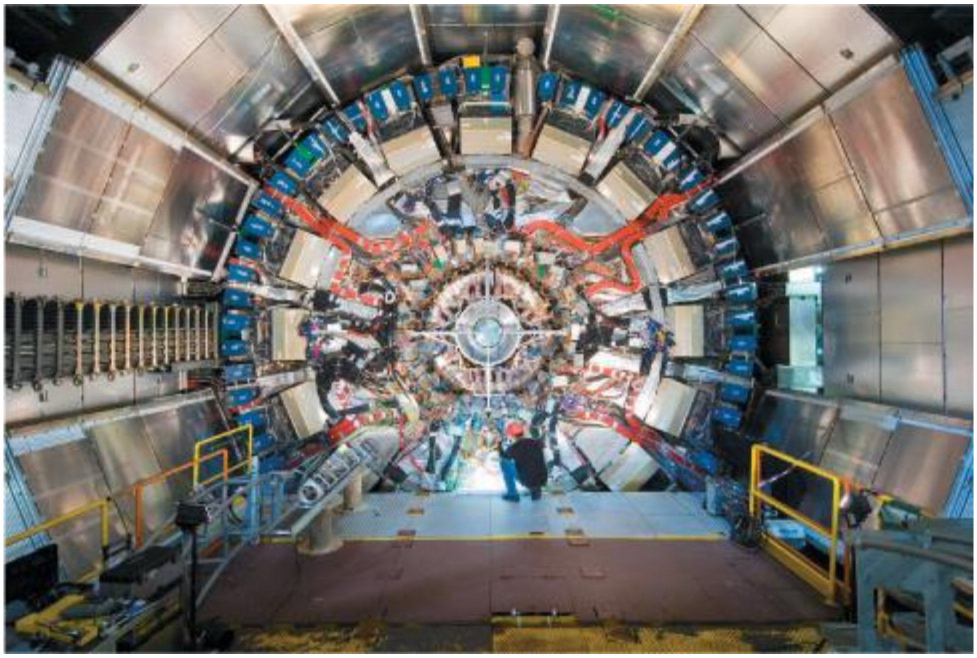
$$R = R_0 e^{-\lambda t}, \tag{44-2}$$

and the half-life $T_{1/2}$, decay constant λ, and mean life τ are related by Eq. 42-18,

$$T_{1/2} = \frac{\ln 2}{\lambda} = \tau \ln 2. \tag{44-3}$$

The half-lives of the new particles range from about 10^{-6} s to 10^{-23} s. Indeed, some of the particles last so briefly that they cannot be detected directly but can only be inferred from indirect evidence.

These new particles are commonly produced in head-on collisions between protons or electrons accelerated to high energies in accelerators at places like Brookhaven National Laboratory (on Long Island, New York), Fermilab (near Chicago), CERN (near Geneva, Switzerland), SLAC (at Stanford University in California), and DESY (near Hamburg, Germany). They are discovered with particle detectors that have grown in sophistication until they rival the size and complexity of entire accelerators of only a few decades ago.



One of the detectors at the Large Hadron Collider at CERN, where the Standard Model of the elementary particles will be put to the test.
(© CERN Geneva)

Today there are several hundred known particles. Naming them has strained the resources of the Greek alphabet, and most are known only by an assigned number in a periodically issued compilation. To make sense of this array of particles, we look for simple physical criteria by which we can place the particles in categories. The result is known as the **Standard Model** of particles. Although this model is continuously challenged by theorists, it remains our best scheme of understanding all the particles discovered to date.

To explore the Standard Model, we make the following three rough cuts among the known particles: fermion or boson, hadron or lepton, particle or antiparticle? Let's now look at the categories one by one.

Fermion or Boson?

All particles have an intrinsic angular momentum called **spin,** as we discussed for electrons, protons, and neutrons in Section 32-7. Generalizing the notation of that section, we can write the component of spin $\vec{S}$ in any direction (assume the component to be along a *z* axis) as

$$S_z = m_s\hbar \quad \text{for } m_s = s,\ s-1, \ldots,\ -s, \tag{44-4}$$

in which $\hbar$ is $h/2\pi$, m_s is the *spin magnetic quantum number*, and *s* is the *spin quantum number*. This last can have either positive half-integer values $\left(\frac{1}{2}, \frac{3}{2}, \ldots\right)$ or nonnegative integer values (0, 1, 2, …). For example, an electron has the value $s = \frac{1}{2}$. Hence the spin of an electron (measured along any direction, such as the *z* direction) can have the values

$$S_z = \frac{1}{2}\hbar \quad \text{(spin up)}$$

or

$$S_z = -\frac{1}{2}\hbar \quad \text{(spin down)}.$$

Confusingly, the term *spin* is used in two ways: It properly means a particle's intrinsic angular momentum $\vec{S}$, but it is often used loosely to mean the particle's spin quantum number *s*. In the latter case, for example, an electron is said to be a spin-$\frac{1}{2}$ particle.

Particles with half-integer spin quantum numbers (like electrons) are called **fermions,** after Fermi, who (simultaneously with Paul Dirac) discovered the statistical laws that govern their behavior. Like electrons, protons and neutrons also have $s = \frac{1}{2}$ and are fermions.

Particles with zero or integer spin quantum numbers are called **bosons**, after Indian physicist Satyendra Nath Bose, who (simultaneously with Albert Einstein) discovered the governing statistical laws for *those* particles. Photons, which have $s = 1$, are bosons; you will soon meet other particles in this class.

This may seem a trivial way to classify particles, but it is very important for this reason:

Fermions obey the Pauli exclusion principle, which asserts that only a single particle can be assigned to a given quantum state. Bosons *do not* obey this principle. Any number of bosons can occupy a given quantum state.

We saw how important the Pauli exclusion principle is when we “built up” the atoms by assigning (spin-$\frac{1}{2}$) electrons to individual quantum states. Using that principle led to a full accounting of the structure and properties of atoms of different types and of solids such as metals and semiconductors.

Because bosons do *not* obey the Pauli principle, those particles tend to pile up in the quantum state of lowest energy. In 1995 a group in Boulder, Colorado, succeeded in producing a condensate of about 2000 rubidium-87 atoms—they are bosons—in a single quantum state of approximately zero energy.

For this to happen, the rubidium has to be a vapor with a temperature so low and a density so great that the de Broglie wavelengths of the individual atoms are greater than the average separation between the atoms. When this condition is met, the wave functions of the individual atoms overlap and the entire assembly becomes a single quantum system (one big atom) called a *Bose–Einstein condensate.* Figure 44-1 shows that, as the temperature of the rubidium vapor is lowered to about 1.70×10^{-7} K, the atoms do indeed “collapse” into a single sharply defined state corresponding to approximately zero speed.

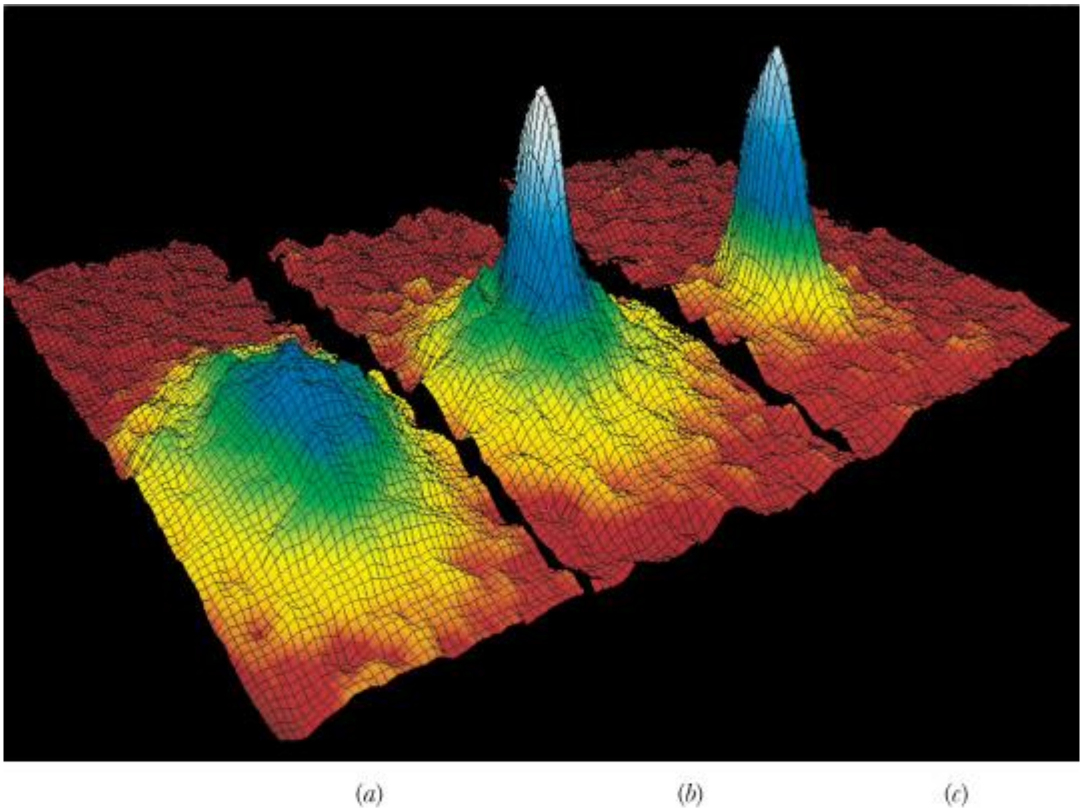


Figure 44-1 Three plots of the particle speed distribution in a vapor of rubidium-87 atoms. The temperature of the vapor is successively reduced from plot (*a*) to plot (*c*). Plot (*c*) shows a sharp peak centered around zero speed; that is, all the atoms are in the same quantum state. The achievement of such a Bose–Einstein condensate, often called the Holy Grail of atomic physics, was finally recorded in 1995.
(Courtesy Michael Mathews)

Hadron or Lepton?

We can also classify particles in terms of the four fundamental forces that act on them. The *gravitational force* acts on *all* particles, but its effects at the level of subatomic particles are so weak that we need not consider that force (at least not in today's research). The *electromagnetic force* acts on all *electrically charged* particles; its effects are well known, and we can take them into account when we need to; we largely ignore this force in this chapter.

We are left with the *strong force,* which is the force that binds nucleons together, and the *weak force,* which is involved in beta decay and similar processes. The weak force acts on all particles, the strong force only on some.

We can, then, roughly classify particles on the basis of whether the strong force acts on them. Particles on which the strong force acts are called **hadrons.** Particles on which the strong force does *not* act, leaving the weak force as the dominant force, are called **leptons.** Protons, neutrons, and pions are hadrons; electrons and neutrinos are leptons.

We can make a further distinction among the hadrons because some of them are bosons (we call them **mesons**); the pion is an example. The other hadrons are fermions (we call them **baryons**); the proton is an example.

Particle or Antiparticle?

In 1928 Dirac predicted that the electron e^- should have a positively charged counterpart of the same mass and spin. The counterpart, the *positron* e^+, was discovered in cosmic radiation in 1932 by Carl Anderson. Physicists then gradually realized that *every* particle has a corresponding **antiparticle.** The members of such pairs have the same mass and spin but opposite signs of electric charge (if they are charged) and opposite signs of quantum numbers that we have not yet discussed.

At first, *particle* was used to refer to the common particles such as electrons, protons, and neutrons, and *antiparticle* referred to their rarely detected counterparts. Later, for the less common particles, the assignment of *particle* and *antiparticle* was made so as to be consistent with certain conservation laws that we shall discuss later in this chapter. (Confusingly, both particles and antiparticles are sometimes called particles when no distinction is needed.) We often, but not always, represent an antiparticle by putting a bar over the symbol for the particle. Thus, p is the symbol for the proton, and $\bar{p}$ (pronounced "p bar") is the symbol for the antiproton.

When a particle meets its antiparticle, the two can *annihilate* each other. That is, the particle and antiparticle disappear and their combined energies reappear in other forms. For an electron annihilating with a positron, this energy reappears as two gamma-ray photons:

$$e^- + e^+ \rightarrow \gamma + \gamma. \tag{44-5}$$

If the electron and positron are stationary when they annihilate, their total energy is their total mass energy, and that energy is then shared equally by the two photons. To conserve momentum and because photons cannot be stationary, the photons fly off in opposite directions.

Large numbers of antihydrogen atoms (each with an antiproton and positron instead of a proton and electron in a hydrogen atom) are now being manufactured and studied at CERN. The Standard Model predicts that a transition in an antihydrogen atom (say, between the first excited state and the ground state) is identical to the same transition in a hydrogen atom. Thus, any difference in the transitions would clearly signal that the Standard Model is erroneous; no difference has yet been spotted.

An assembly of antiparticles, such as an antihydrogen atom, is often called *antimatter* to distinguish it from an assembly of common particles (*matter*). (The terms can easily be confusing when the word "matter" is used to describe anything that has mass.) We can speculate that future scientists and engineers may construct objects of antimatter. However, no evidence suggests that nature has already done this on an astronomical scale because all stars and galaxies appear to consist largely of matter and not antimatter. This is a perplexing observation because it means that when the universe began, some feature biased the conditions toward matter and away from antimatter. (For example, electrons are common but positrons are not.) This bias is still not well understood.

Before pressing on with the task of classifying the particles, let us step aside for a moment and capture some of the spirit of particle research by analyzing a typical particle event—namely, that shown in the bubble-chamber photograph of Fig. 44-2*a*.

Bubble chamber photograph

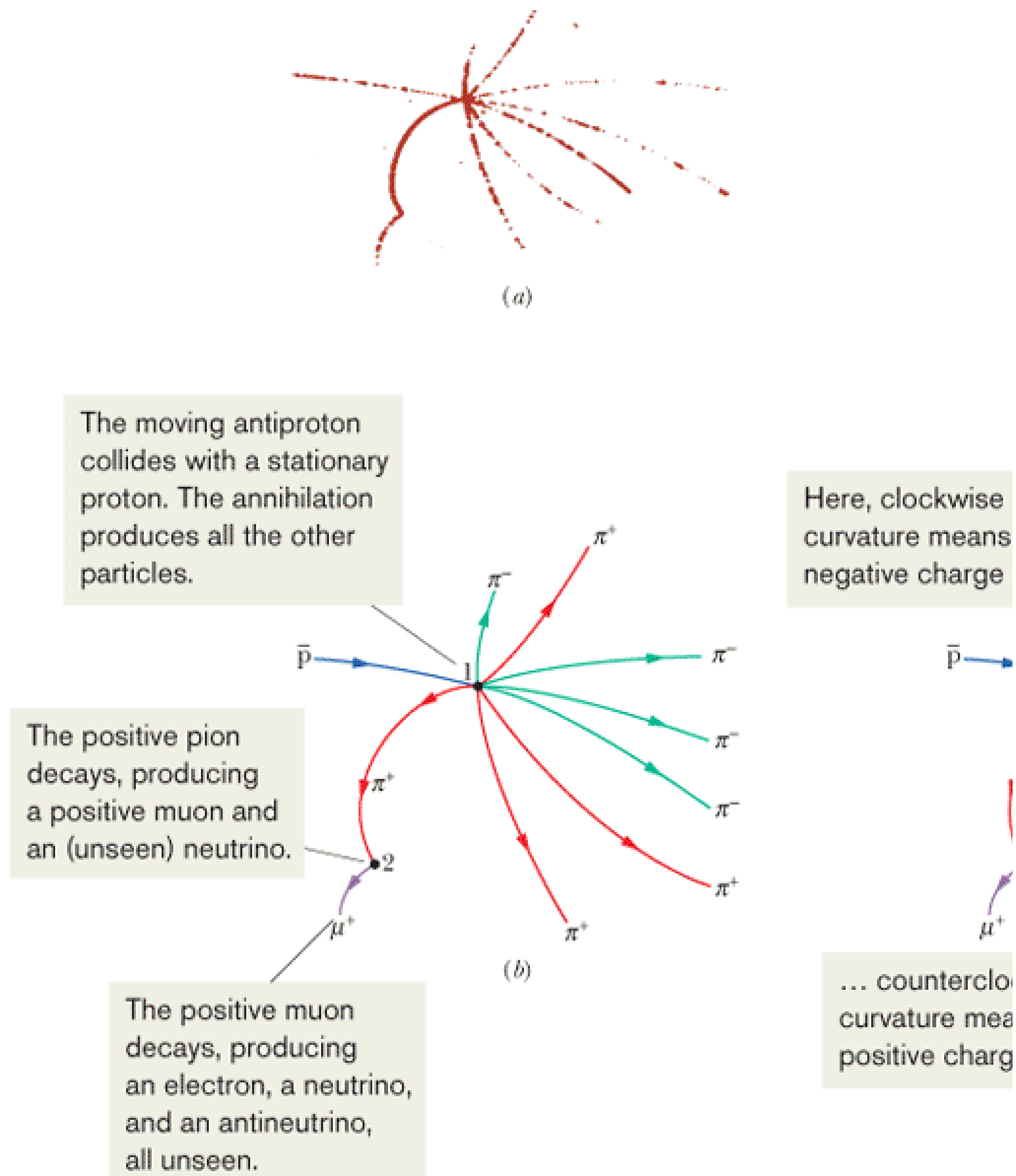


Figure 44-2(*a*) A bubble-chamber photograph of a series of events initiated by an antiproton that enters the chamber from the left. (*b*) The tracks redrawn and labeled for clarity. (*c*) The tracks are curved because a magnetic field present in the chamber

exerts a deflecting force on each moving charged particle.
(Courtesy Lawrence Berkeley Laboratory)

The tracks in this figure consist of bubbles formed along the paths of electrically charged particles as they move through a chamber filled with liquid hydrogen. We can identify the particle that makes a particular track by—among other means—measuring the relative spacing between the bubbles. The chamber lies in a uniform magnetic field that deflects the tracks of positively charged particles counterclockwise and the tracks of negatively charged particles clockwise. By measuring the radius of curvature of a track, we can calculate the momentum of the particle that made it. Table 44-1 shows some properties of the particles and antiparticles that participated in the event of Fig. 44-2*a*, including those that did not make tracks. Following common practice, we express the masses of the particles listed in Table 44-1—and in all other tables in this chapter—in the unit MeV/c^2. The reason for this notation is that the rest energy of a particle is needed more often than its mass. Thus, the mass of a proton is shown in Table 44-1 to be 938.3 MeV/c^2. To find the proton's rest energy, multiply this mass by c^2 to obtain 938.3 MeV.

Table 44-1 The Particles or Antiparticles Involved in the Event of Fig. 44-2

Particle	Symbol	Charge q	Mass (MeV/c^2)	Spin Quantum Number s	Identity	Mean Life (s)	Antiparticle
Neutrino	ν	0	$\approx 1 \times 10^{-7}$	$\frac{1}{2}$	Lepton	Stable	$\bar{\nu}$
Electron	e^-	-1	0.511	$\frac{1}{2}$	Lepton	Stable	e^+
Muon	μ^-	-1	105.7	$\frac{1}{2}$	Lepton	2.2×10^{-6}	μ^+
Pion	π^+	+1	139.6	0	Meson	$2.6\ 10^{-8}$	π^-
Proton	p	+1	938.3	$\frac{1}{2}$	Baryon	Stable	$\bar{p}$

The general tools used for the analysis of photographs like Fig. 44-2*a* are the laws of conservation of energy, linear momentum, angular momentum, and electric charge, along with other conservation laws that we have not yet discussed. Figure 44-2*a* is actually one of a stereo pair of photographs so that, in practice, these analyses are carried out in three dimensions.

The event of Fig. 44-2*a* is triggered by an energetic antiproton $(\bar{p})$that, generated in an accelerator at the Lawrence Berkeley Laboratory, enters the chamber from the left. There are three separate subevents; one occurs at point 1 in Fig. 44-2*b*, the second occurs at point 2, and the third occurs out of the frame of the figure. Let's examine each:

1. ***Proton–Antiproton Annihilation.*** At point 1 in Fig. 44-2*b*, the initiating antiproton (blue track) slams into a proton of the liquid hydrogen in the chamber, and the result is mutual annihilation. We can tell that annihilation occurred while the incoming antiproton was in flight because most of the particles generated in the encounter move in the forward direction—that is, toward the right in Fig. 44-2. From the principle of conservation of linear momentum, the incoming antiproton must have had a forward momentum when it underwent annihilation. Further, because the particles are charged and moving through a magnetic field, the curvature of the paths reveal whether the particles are negatively charged (like the incident antiproton) or positively charged

(Fig. 44-2*c*).
The total energy involved in the collision of the antiproton and the proton is the sum of the antiproton's kinetic energy and the two (identical) rest energies of those two particles (2 × 938.3 MeV, or 1876.6 MeV). This is enough energy to create a number of lighter particles and give them kinetic energy. In this case, the annihilation produces four positive pions (red tracks in Fig. 44-2*b*) and four negative pions (green tracks). (For simplicity, we assume that no gamma-ray photons, which would leave no tracks because they lack electric charge, are produced.) Thus we conclude that the annihilation process is

$$\mathrm{p} + \overline{\mathrm{p}} \rightarrow 4\pi^{+} + 4\pi^{-}. \qquad (44\text{-}6)$$

We see from Table 44-1 that the positive pions (π^+) are *particles* and the negative pions (π^-) are *antiparticles.* The reaction of Eq. 44-6 is a *strong interaction* (it involves the strong force) because all the particles involved are hadrons.

Let us check whether electric charge is conserved in the reaction. To do so, we can write the electric charge of a particle as qe, in which q is a **charge quantum number.** Then determining whether electric charge is conserved in a process amounts to determining whether the initial net charge quantum number is equal to the final net charge quantum number. In the process of Eq. 44-6, the initial net charge number is 1 + (-1), or 0, and the final net charge number is 4(1) + 4(-1), or 0. Thus, charge *is* conserved.

For the energy balance, note from above that the energy available from the p-$\overline{\mathrm{p}}$ annihilation process is at least the sum of the proton and antiproton rest energies, 1876.6 MeV. The rest energy of a pion is 139.6 MeV, which means the rest energies of the eight pions amount to 8 × 139.6 MeV, or 1116.8 MeV. This leaves at least about 760 MeV to distribute among the eight pions as kinetic energy. Thus, the requirement of energy conservation is easily met.

2. ***Pion Decay.*** Pions are unstable particles and decay with a mean lifetime of 2.6×10^{-8} s. At point 2 in Fig. 44-2*b*, one of the positive pions comes to rest in the chamber and decays spontaneously into an antimuon μ^+ (purple track) and a neutrino ν:

$$\pi^{+} \rightarrow \mu^{+} + \nu. \qquad (44\text{-}7)$$

The neutrino, being uncharged, leaves no track. Both the antimuon and the neutrino are leptons; that is, they are particles on which the strong force does not act. Thus, the decay process of Eq. 44-7, which is governed by the weak force, is described as a *weak interaction.*
Let's consider the energies in the decay. From Table 44-1, the rest energy of an antimuon is 105.7 MeV and the rest energy of a neutrino is approximately 0. Because the pion is at rest when it decays, its energy is just its rest energy, 139.6 MeV. Thus, an energy of 139.6 MeV - 105.7 MeV, or 33.9 MeV, is available to share between the antimuon and the neutrino as kinetic energy.

Let us check whether spin angular momentum is conserved in the process of Eq. 44-7. This amounts to determining whether the net component S_z of spin angular momentum along some arbitrary z axis can be conserved by the process. The spin quantum numbers s of the particles in the process are 0 for the pion π^+ and $\frac{1}{2}$ for both the antimuon μ^+ and the neutrino ν. Thus, for π^+, the component S_z must be $0\hbar$, and for μ^+ and ν, it can be either $+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$.

The net component S_z is conserved by the process of Eq. 44-7 if there is *any* way in which the initial $S_z(=0\hbar)$ can be equal to the final net S_z. We see that if one of the products, either μ^+ or ν, has $S_z = +\frac{1}{2}\hbar$ and the other has $S_z = -\frac{1}{2}\hbar$, then their final net value is $0\hbar$. Thus, because S_z can be conserved, the decay process of Eq. 44-7 *can* occur.

From Eq. 44-7, we also see that the net charge is conserved by the process: before the process the net charge quantum number is +1, and after the process it is +1 + 0 = +1.

3. ***Muon Decay.*** Muons (whether μ^- or μ^+) are also unstable, decaying with a mean life of 2.2×10^{-6} s. Although the decay products are not shown in Fig. 44-2, the antimuon produced in the reaction of Eq. 44-7 comes to rest and decays spontaneously according to

$$\mu^+ \rightarrow e^+ + \nu + \bar{\nu}. \tag{44-8}$$

The rest energy of the antimuon is 105.7 MeV, and that of the positron is only 0.511 MeV, leaving 105.2 MeV to be shared as kinetic energy among the three particles produced in the decay process of Eq. 44-8.

You may wonder: Why *two* neutrinos in Eq. 44-8? Why not just one, as in the pion decay in Eq. 44-7? One answer is that the spin quantum numbers of the antimuon, the positron, and the neutrino are each $\frac{1}{2}$; with only one neutrino, the net component S_z of spin angular momentum could not be conserved in the antimuon decay of Eq. 44-8. In Section 44-4 we shall discuss another reason.

Momentum and kinetic energy in a pion decay

A stationary positive pion can decay according to

$$\pi^+ \rightarrow \mu^+ + \nu.$$

What is the kinetic energy of the antimuon μ^+? What is the kinetic energy of the neutrino ν?

KEY IDEA

The pion decay process must conserve both total energy and total linear momentum.

Energy conservation:

Let us first write the conservation of total energy (rest energy mc^2 plus kinetic energy K) for the decay process as

$$m_\pi c^2 + K_\pi = m_\pi c^2 + K_\mu + m_\nu c^2 + K_\nu.$$

Because the pion was stationary, its kinetic energy K_π is zero. Then, using the masses listed for m_π, m_μ, and m_ν in Table 44-1, we find

$$\begin{aligned} K_\mu + K_\nu &= m_\pi c^2 - m_\mu c^2 - m_\nu c^2 \\ &= 139.6\,\mathrm{MeV} - 105.7\,\mathrm{MeV} - 0 \\ &= 33.9\,\mathrm{MeV}, \end{aligned} \tag{44-9}$$

where we have approximated m_ν as zero.

Momentum conservation: We cannot solve Eq. 44-9 for either K_μ or K_ν separately, and so let us next apply the principle of conservation of linear momentum to the decay process. Because the pion is stationary when it decays, that principle requires that the muon and neutrino move in opposite directions after the decay. Assume that their motion is along an axis. Then, for components along that axis, we can write the conservation of linear momentum for the decay as

$$p_\pi = p_\mu + p_\nu,$$

which, with $p_\pi = 0$, gives us

$$p_\mu = -p_\nu. \tag{44-10}$$

Relating p and K: We want to relate these momenta p_μ and $-p_\nu$ to the kinetic energies K_μ and K_ν so that we can solve for the kinetic energies. Because we have no reason to believe that classical physics can be applied, we use Eq. 37-54, the momentum–kinetic energy relation from special relativity:

$$(pc)^2 = K^2 + 2Kmc^2. \tag{44-11}$$

From Eq. 44-10, we know that

$$(p_\mu c)^2 = (p_\nu c)^2. \tag{44-12}$$

Substituting from Eq. 44-11 for each side of Eq. 44-12 yields

$$K_\mu^2 + 2K_\mu m_\mu c^2 = K_\nu^2 + 2K_\nu m_\nu c^2.$$

Approximating the neutrino mass to be $m_\nu = 0$, substituting $K_\nu = 33.9$ MeV - K_μ from Eq. 44-9, and then solving for K_μ, we find

$$\begin{aligned} K_\mu &= \frac{(33.9\,\mathrm{MeV})^2}{(2)\left(33.9\,\mathrm{MeV} + m_\mu c^2\right)} \\ &= \frac{(33.9\,\mathrm{MeV})^2}{(2)(33.9\,\mathrm{MeV} + 105.7\,\mathrm{MeV})} \\ &= 4.12\,\mathrm{MeV}. \end{aligned} \qquad \text{(Answer)}$$

The kinetic energy of the neutrino is then, from Eq. 44-9,

$$\begin{aligned} K_\nu &= 33.9\,\mathrm{MeV} - K_\mu = 33.9\,\mathrm{MeV} - 4.12\,\mathrm{MeV} \\ &= 29.8\,\mathrm{MeV}. \end{aligned} \qquad \text{(Answer)}$$

We see that, although the magnitudes of the momenta of the two recoiling particles are the same, the neutrino gets the larger share (88%) of the kinetic energy.

Q in a proton-pion reaction

The protons in the material filling a bubble chamber are bombarded with energetic antiparticles known as negative pions. At collision points, a proton and a pion transform into a negative kaon and a positive sigma:

$$\pi^- + \mathrm{p} \rightarrow \mathrm{K}^- + \Sigma^+.$$

The rest energies of these particles are

π^-	139.6 MeV	K^-	493.7 MeV
p	938.3 MeV	Σ^+	1189.4 MeV

What is the Q of the reaction?

KEY IDEA

The Q of a reaction is

$$Q = \begin{pmatrix}\text{intial total}\\ \text{mass energy}\end{pmatrix} - \begin{pmatrix}\text{final total}\\ \text{mass energy}\end{pmatrix}.$$

Calculation:

For the given reaction, we find

$$\begin{aligned} Q &= \left(m_\pi c^2 + m_p c^2\right) - \left(m_K c^2 + m_\Sigma c^2\right) \\ &= (139.6\ \text{MeV} + 938.3\ \text{MeV}) - (493.7\ \text{MeV} + 1189.4\ \text{MeV}) \\ &= -605\ \text{MeV}. \end{aligned}$$

(Answer)

The minus sign means that the reaction is *endothermic;* that is, the incoming pion (π^-) must have a kinetic energy greater than a certain threshold value if the reaction is to occur. The threshold energy is actually greater than 605 MeV because linear momentum must be conserved. (The incoming pion has momentum.) This means that the kaon (K^-) and the sigma (Σ^+) not only must be created but also must be given some kinetic energy. A relativistic calculation whose details are beyond our scope shows that the threshold energy for the reaction is 907 MeV.

44-4 The Leptons

In this and the next section, we discuss some of the particles of one of our classification schemes: lepton or hadron. We begin with the leptons, those particles on which the strong force does *not* act. So far, we have encountered the familiar electron and the neutrino that accompanies it in beta decay. The muon, whose decay is described in Eq. 44-8, is another member of this family. Physicists gradually learned that the neutrino that appears in Eq. 44-7, associated with the production of a muon, is *not the same particle* as the neutrino produced in beta decay, associated with the appearance of an electron. We call the former the **muon neutrino** (symbol ν_μ) and the latter the **electron neutrino** (symbol ν_e) when it is necessary to distinguish between them.

These two types of neutrino are known to be different particles because, if a beam of muon neutrinos (produced from pion decay as in Eq. 44-7) strikes a solid target, *only muons*—and never electrons—are produced. On the other hand, if electron neutrinos (produced by the beta decay of fission products in a nuclear reactor) strike a solid target, *only electrons*—and never muons—are produced.

Another lepton, the **tau,** was discovered at SLAC in 1975; its discoverer, Martin Perl, shared the 1995 Nobel Prize in physics. The tau has its own associated neutrino, different still from the other two. Table 44-2 lists all the leptons (both particles and antiparticles); all have a spin quantum number *s* of $\frac{1}{2}$.

Table 44-2 The Leptons [a]

Family	Particle	Symbol	Mass (MeV/c^2)	Charge q	Antiparticle
Electron	Electron	e^-	0.511	-1	e^+
	Electron neutrino[b]	ν_e	$\approx 1 \times 10^{-7}$	0	$\bar{\nu}_e$
Muon	Muon	μ^-	105.7	-1	μ^+
	Muon neutrino[b]	ν_μ	$\approx 1 \times 10^{-7}$	0	$\bar{\nu}_\mu$
Tau	Tau	τ^-	1777	-1	τ^+
	Tau neutrino[b]	ν_τ	$\approx 1 \times 10^{-7}$	0	$\bar{\nu}_\tau$

a) All leptons have spin quantum numbers of $\frac{1}{2}$ and are thus fermions.

b)The neutrino masses have not been well determined.

There are reasons for dividing the leptons into three families, each consisting of a particle (electron, muon, or tau), its associated neutrino, and the corresponding antiparticles. Furthermore, there are reasons to believe that there are *only* the three families of leptons shown in Table 44-2. Leptons have no internal structure and no measurable dimensions; they are believed to be truly pointlike fundamental particles when they interact with other particles or with electromagnetic waves.

The Conservation of Lepton Number

According to experiment, particle interactions involving leptons obey a conservation law for a quantum number called the **lepton number** L. Each (normal) particle in Table 44-2 is assigned $L = +1$, and each antiparticle is assigned $L = -1$. All other particles, which are not leptons, are assigned $L = 0$. Also according to experiment,

> In all particle interactions, the net lepton number *for each family* is separately conserved.

Thus, there are actually three lepton numbers L_e, L_μ, L_τ, and the net of *each* must remain unchanged during any particle interaction. This experimental fact is called the law of **conservation of lepton number.**

We can illustrate this law by reconsidering the antimuon decay process shown in Eq. 44-8, which we now write more fully as

$$\mu^{+} \rightarrow e^{+} + \nu_{e} + \bar{\nu}_{\mu}. \tag{44-13}$$

Consider this first in terms of the muon family of leptons. The μ^+ is an antiparticle (see Table 44-2) and thus has the muon lepton number $L_\mu = -1$. The two particles e^+ and ν_e do not belong to the muon family and thus have $L_\mu = 0$. This leaves $\bar{\nu}_\mu$ on the right which, being an antiparticle, also has the muon lepton number $L_\mu = -1$. Thus, both sides of Eq. 44-13 have the same net muon lepton number—namely, $L_\mu = -1$; if they did not, the μ^+ would not decay by this process.

No members of the electron family appear on the left in Eq. 44-13; so there the net electron lepton number must be $L_e = 0$. On the right side of Eq. 44-13, the positron, being an antiparticle (again see Table 44-2), has the electron lepton number $L_e = -1$. The electron neutrino ν_e, being a particle, has the electron number $L_e = +1$. Thus, the net electron lepton number for these two particles on the right in Eq. 44-13 is also zero; the electron lepton number is also conserved in the process.

CHECKPOINT 1

(a) The π^+ meson decays by the process $\pi^+ \rightarrow \mu^+ + \nu$. To what lepton family does the neutrino ν belong? (b) Is this neutrino a particle or an antiparticle? (c) What is its lepton number?

Answer:

(a) the muon family; (b) a particle; (c) $L_\mu = +1$

Because no members of the tau family appear on either side of Eq. 44-13, we must have $L_\tau = 0$ on each side. Thus, each of the lepton quantum numbers L_μ, L_e, and L_τ remains unchanged during the decay process of Eq. 44-13, their constant values being -1, 0, and 0, respectively. This example is but one illustration of the conservation of lepton number; this law holds for all particle interactions. However, note that the law is based on (countless) experimental observations. We do not know *why* the law must be obeyed; we only know that this conservation law is part of the way our universe works.

44-5 The Hadrons

We are now ready to consider hadrons (baryons and mesons), those particles whose interactions are governed by the strong force. We start by adding another conservation law to our list: conservation of baryon number.

To develop this conservation law, let us consider the proton decay process

$$p \rightarrow e^{+} + \nu_e. \qquad (44\text{-}14)$$

This process *never* happens. We should be glad that it does not because otherwise all protons in the universe would gradually change into positrons, with disastrous consequences for us. Yet this decay process does not violate the conservation laws involving energy, linear momentum, or lepton number.

We account for the apparent stability of the proton—and for the absence of many other processes that might otherwise occur—by introducing a new quantum number, the **baryon number** *B*, and a new conservation law, the **conservation of baryon number:**

To every baryon we assign $B = +1$. To every antibaryon we assign $B = -1$. To all particles of other types we assign $B = 0$. A particle process cannot occur if it changes the net baryon number.

In the process of Eq. 44-14, the proton has a baryon number of $B = +1$ and the positron and neutrino both have a baryon number of $B = 0$. Thus, the process does not conserve baryon number and cannot occur.

CHECKPOINT 2

This mode of decay for a neutron is *not* observed:

$$n \rightarrow p + e^{-}.$$

Which of the following conservation laws does this process violate: (a) energy, (b) angular momentum, (c) linear momentum, (d) charge, (e) lepton number, (f) baryon number? The masses are $m_n = 939.6\ \text{MeV}/c^2$, $m_p = 938.3\ \text{MeV}/c^2$, and $m_e = 0.511\ \text{MeV}/c^2$.

Answer:

b and e

Proton decay: conservation of quantum numbers, energy, and momentum

Determine whether a stationary proton can decay according to the scheme

$$p \rightarrow \pi^0 + \pi^+.$$

Properties of the proton and the π^+ pion are listed in Table 44-1. The π^0 pion has zero charge, zero spin, and a mass energy of 135.0 MeV.

KEY IDEA

We need to see whether the proposed decay violates any of the conservation laws we have discussed.

Electric charge:

We see that the net charge quantum number is initially +1 and finally 0 + 1, or +1. Thus, charge is conserved by the decay. Lepton number is also conserved, because none of the three particles is a lepton and thus each lepton number is zero.

Linear momentum: Because the proton is stationary, with zero linear momentum, the two pions must merely move in opposite directions with equal magnitudes of linear momentum (so that their total linear momentum is also zero) to conserve linear momentum. The fact that linear momentum *can* be conserved means that the process does not violate the conservation of linear momentum.

Energy: Is there energy for the decay? Because the proton is stationary, that question amounts to asking whether the proton's mass energy is sufficient to produce the mass energies and kinetic energies of the pions. To answer, we evaluate the Q of the decay:

$$\begin{aligned} Q &= \begin{pmatrix} \text{intial total} \\ \text{mass energy} \end{pmatrix} - \begin{pmatrix} \text{final total} \\ \text{mass energy} \end{pmatrix} \\ &= m_{\mathrm{p}}c^2 - \left(m_0c^2 + m_+c^2\right) \\ &= 938.3\ \mathrm{MeV} - (135.0\ \mathrm{MeV} + 139.6\ \mathrm{MeV}) \\ &= (663.7\ \mathrm{MeV.}) \end{aligned}$$

The fact that Q is positive indicates that the initial mass energy exceeds the final mass energy. Thus, the proton *does* have enough mass energy to create the pair of pions.

Spin: Is spin angular momentum conserved by the decay? This amounts to determining whether the net component S_z of spin angular momentum along some arbitrary z axis can be conserved by the decay. The spin quantum numbers s of the particles in the process are $\frac{1}{2}$ for the proton and 0 for both pions. Thus, for the proton the component S_z can be either $+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$ and for each pion it is $0\hbar$. We see that there is no way that S_z can be conserved. Hence, spin angular momentum is not conserved, and the proposed decay of the proton cannot occur.

Baryon number: The decay also violates the conservation of baryon number: The proton has a baryon number of $B = +1$, and both pions have a baryon number of $B = 0$. Thus, nonconservation of baryon number is another reason the proposed decay cannot occur.

Xi-minus decay: conservation of quantum numbers

A particle called xi-minus and having the symbol Ξ^- decays as follows:

$$\Xi^- \rightarrow \Lambda^0 + \pi^-.$$

The Λ^0 particle (called lambda-zero) and the π^- particle are both unstable. The following decay processes occur in *cascade* until only relatively stable products remain:

$$\Lambda^0 \longrightarrow p + \pi^- \quad \pi^- \longrightarrow \mu^- + \bar{\nu}_\mu$$
$$\mu^- \longrightarrow e^- + \nu_\mu + \bar{\nu}_e.$$

(a) Is the Ξ^- particle a lepton or a hadron? If the latter, is it a baryon or a meson?

KEY IDEAS

(1) Only three families of leptons exist (Table 44-2) and none include the Ξ^- particle. Thus, the Ξ^- must be a hadron. (2) To answer the second question we need to determine the baryon number of the Ξ^- particle. If it is +1 or -1, then the Ξ^- is a baryon. If, instead, it is 0, then the Ξ^- is a meson.

Baryon number:

To see, let us write the overall decay scheme, from the initial Ξ^- to the final relatively stable products, as

$$\Xi^- \rightarrow p + 2(e^- + \bar{\nu}_e) + 2(\nu_\mu + \bar{\nu}_\mu). \qquad (44\text{-}15)$$

On the right side, the proton has a baryon number of +1 and each electron and neutrino has a baryon number of 0. Thus, the net baryon number of the right side is +1. That must then be the baryon number of the lone Ξ^- particle on the left side. We conclude that the Ξ^- particle is a baryon.

(b) Does the decay process conserve the three lepton numbers?

KEY IDEA

Any process must separately conserve the net lepton number for each lepton family of Table 44-2.

Lepton number:

Let us first consider the electron lepton number L_e, which is +1 for the electron e^-, -1 for the anti-electron neutrino $\bar{\nu}_e$, and 0 for the other particles in the overall decay of Eq. 44-15. We see that the net L_e is 0 before the decay and 2[+1 + (-1)] + 2(0 + 0) = 0 after the decay. Thus, the net electron lepton number *is* conserved. You can similarly show that the net muon lepton number and the net tau lepton number are also conserved.

(c) What can you say about the spin of the Ξ^- particle?

KEY IDEA

The overall decay scheme of Eq. 44-15 must conserve the net spin component S_z.

Spin:

We can determine the spin component S_z of the Ξ^- particle on the left side of Eq. 44-15 by considering the S_z components of the nine particles on the right side. All nine of those particles are spin-$\frac{1}{2}$ particles and thus can have S_z of either $+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$. No matter how we choose between those two possible values of S_z, the net S_z for those nine particles must be a *half-integer* times $\hbar$. Thus, the Ξ^- particle must have S_z of a *half-integer* times $\hbar$, and that means that its spin quantum number s must be a half-integer. (It is $\frac{1}{2}$.)

44-6 Still Another Conservation Law

Particles have intrinsic properties in addition to the ones we have listed so far: mass, charge, spin, lepton number, and baryon number. The first of these additional properties was discovered when researchers observed that certain new particles, such as the kaon (K) and the sigma (Σ), always seemed to be produced in pairs. It seemed impossible to produce only one of them at a time. Thus, if a beam of energetic pions interacts with the protons in a bubble chamber, the reaction

$$\pi^+ + \mathrm{p} \rightarrow \mathrm{K}^+ + \Sigma^+ \qquad (44\text{-}16)$$

often occurs. The reaction

$$\pi^+ + \mathrm{p} \rightarrow \pi^+ + \Sigma^+, \qquad (44\text{-}17)$$

which violates no conservation law known in the early days of particle physics, never occurs.

It was eventually proposed (by Murray Gell-Mann in the United States and independently by K. Nishijima in Japan) that certain particles possess a new property, called **strangeness,** with its own quantum number S and its own conservation law. (Be careful not to confuse the symbol S here with spin.) The name *strangeness* arises from the fact that, before the identities of these particles were pinned down, they were known as “strange particles,” and the label stuck.

The proton, neutron, and pion have $S = 0$; that is, they are not “strange.” It was proposed, however, that the K^+ particle has strangeness $S = +1$ and that Σ^+ has $S = -1$. In the reaction of Eq. 44-16, the net strangeness is initially zero and finally zero; thus, the reaction conserves strangeness. However, in the reaction shown in Eq. 44-17, the final net strangeness is -1; thus, that reaction does not conserve strangeness and cannot occur. Apparently, then, we must add one more conservation law to our list—the **conservation of strangeness:**

Strangeness is conserved in interactions involving the strong force.

It may seem heavy-handed to invent a new property of particles just to account for a little puzzle like that posed by Eqs. 44-16 and 44-17. However, strangeness soon solved many other puzzles. Still, do not be misled by the whimsical character of the name. Strangeness is no more mysterious a property of particles than is charge. Both are properties that particles may (or may not) have; each is described by an appropriate quantum number. Each obeys a conservation law. Still other properties of particles have been discovered and given even more whimsical names, such as *charm* and *bottomness,* but all are perfectly legitimate properties. Let us see, as an example, how the new property of strangeness "earns its keep" by leading us to uncover important regularities in the properties of the particles.

44-7 The Eightfold Way

There are eight baryons—the neutron and the proton among them—that have a spin quantum number of $\frac{1}{2}$. Table 44-3 shows some of their other properties. Figure 44-3*a* shows the fascinating pattern that emerges if we plot the strangeness of these baryons against their charge quantum number, using a sloping axis for the charge quantum numbers. Six of the eight form a hexagon with the two remaining baryons at its center.

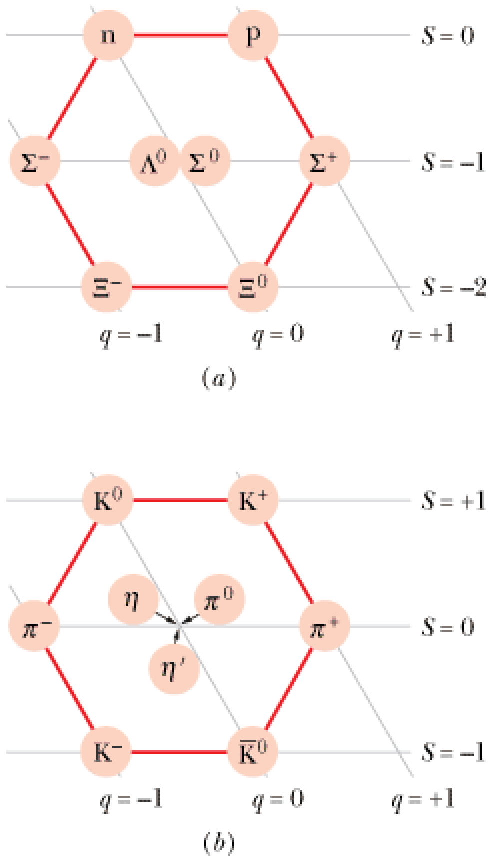


Figure 44-3

(a) The eightfold way pattern for the eight spin-$\frac{1}{2}$ baryons listed in Table 44-3. The particles are represented as disks on a strangeness–charge plot, using a sloping axis for the charge quantum number. (b) A similar pattern for the nine spin-zero mesons listed in Table 44-4.

Table 44-3

Eight Spin-$\frac{1}{2}$ Baryons

			Quantum Numbers	
Particle	Symbol	Mass (MeV/c^2)	Charge q	Strangeness S
Proton	p	938.3	+1	0
Neutron	n	939.6	0	0
Lambda	Λ^0	1115.6	0	-1
Sigma	Σ^+	1189.4	+1	-1
Sigma	Σ^0	1192.5	0	-1
Sigma	Σ^-	1197.3	-1	-1
Xi	Ξ^0	1314.9	0	-2
Xi	Ξ^-	1321.3	-1	-2

Let us turn now from the hadrons called baryons to the hadrons called mesons. Nine with a spin of zero are listed in Table 44-4. If we plot them on a sloping strangeness–charge diagram, as in Fig. 44-3*b*, the same fascinating pattern emerges! These and related plots, called the **eightfold way** patterns,[a] were proposed independently in 1961 by Murray Gell-Mann at the California Institute of Technology and by Yuval Ne'eman at Imperial College, London. The two patterns of Fig. 44-3 are representative of a larger number of symmetrical patterns in which groups of baryons and mesons can be displayed.

Table 44-4 Nine Spin-Zero Mesons*

			Quantum Numbers	
Particle	Symbol	Mass (MeV/c^2)	Charge q	Strangeness S
Pion	π^0	135.0	0	0
Pion	π^+	139.6	+1	0
Pion	π^-	139.6	-1	0
Kaon	K^+	493.7	+1	+1
Kaon	K^-	493.7	-1	-1
Kaon	K^0	497.7	0	+1
Kaon	$\bar{K}^0$	497.7	0	-1
Eta	η	547.5	0	0
Eta prime	η'	957.8	0	0

*)All mesons are bosons, having spins of 0, 1, 2, …. The ones listed here all have a spin of 0.

The symmetry of the eightfold way pattern for the spin-$\frac{3}{2}$ baryons (not shown here) calls for ten particles arranged in a pattern like that of the tenpins in a bowling alley. However, when the pattern was first proposed, only nine such particles were known; the "headpin" was missing. In 1962, guided by theory and the symmetry of the pattern, Gell-Mann made a prediction in which he essentially said:

> *There exists a spin-$\frac{3}{2}$ baryon with a charge of -1, a strangeness of -3, and a rest energy of about 1680 MeV. If you look for this omega minus particle (as I propose to call it), I think you will find it.*

A team of physicists headed by Nicholas Samios of the Brookhaven National Laboratory took up the challenge and found the "missing" particle, confirming all its predicted properties. Nothing beats prompt experimental confirmation for building confidence in a theory!

The eightfold way patterns bear the same relationship to particle physics that the periodic table does to chemistry. In each case, there is a pattern of organization in which vacancies (missing particles or missing elements) stick out like sore thumbs, guiding experimenters in their searches. In the case of the periodic table, its very existence strongly suggests that the atoms of the elements are not fundamental particles but have an underlying structure. Similarly, the eightfold way patterns strongly suggest that the mesons and the baryons must have an underlying structure, in terms of which their properties can be understood. That structure can be explained in terms of the *quark model*, which we now discuss.

44-8 The Quark Model

In 1964 Gell-Mann and George Zweig independently pointed out that the eightfold way patterns can be understood in a simple way if the mesons and the baryons are built up out of subunits that Gell-Mann called **quarks.** We deal first with three of them, called the *up quark* (symbol u), the *down quark* (symbol d), and the *strange quark* (symbol s), and we assign to them the properties displayed in Table 44-5. (The names of the quarks, along with those assigned to three other quarks that we shall meet later, have no meaning other than as convenient labels. Collectively, these names are called the *quark flavors.* We could just as well call them vanilla, chocolate, and strawberry instead of up, down, and strange.)

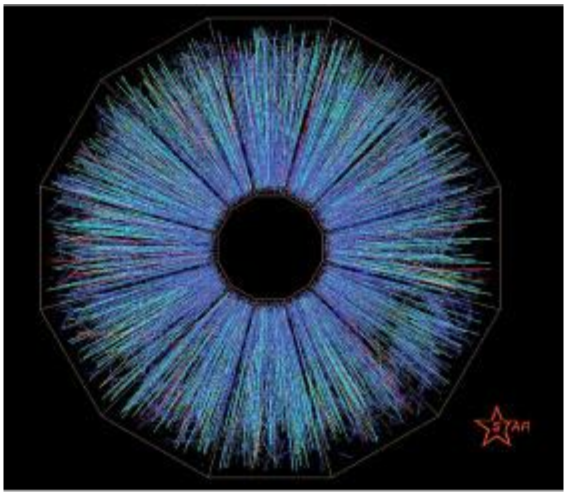
Table 44-5 The Quarks *

			Quantum Numbers			
Particle	Symbol	Mass (MeV/c^2)	Charge q	Strangeness *S*	Baryon Number *B*	Antiparticle
Up	u	5	$+\frac{2}{3}$	0	$+\frac{1}{3}$	$\bar{u}$
Down	d	10	$-\frac{1}{3}$	0	$+\frac{1}{3}$	$\bar{d}$
Charm	c	1500	$+\frac{2}{3}$	0	$+\frac{1}{3}$	$\bar{c}$
Strange	s	200	$-\frac{1}{3}$	-1	$+\frac{1}{3}$	$\bar{s}$
Top	t	175000	$+\frac{2}{3}$	0	$+\frac{1}{3}$	$\bar{t}$
Bottom	b	4300	$-\frac{1}{3}$	0	$+\frac{1}{3}$	$\bar{b}$

*)All quarks (including antiquarks) have spin and thus are fermions. The quantum numbers *q*, *S*, and *B* for each antiquark are the negatives of those for the corresponding quark.

The fractional charge quantum numbers of the quarks may jar you a little. However, withhold judgment until you see how neatly these fractional charges account for the observed integer charges of the mesons and the baryons. In all normal situations, whether here on Earth or in an astronomical process, quarks are always bound up together in twos or threes for reasons that are still not well understood. Such requirements are our normal rule for quark combinations.

An exciting exception to the normal rule occurred in experiments at the RHIC particle collider at the Brookhaven National Laboratory. At the spot where two high-energy beams of gold nuclei collided head-on, the kinetic energy of the particles was so large that it matched the kinetic energy of particles that were present soon after the beginning of the universe (as we discuss in Section 44-14). The protons and neutrons of the gold nuclei were ripped apart to form a momentary gas of individual quarks. (The gas also contained gluons, the particles that normally hold quarks together, as we discuss in Section 44-9.) These experiments at RHIC may be the first time that quarks have been set free of one another since the universe began.



The violent head-on collision of two 30 GeV beams of gold atoms in the RHIC accelerator at the Brookhaven National Laboratory. In the moment of collision, a gas of individual quarks and gluons was created.
(Courtesy Brookhaven National Laboratory)

Quarks and Baryons

Each baryon is a combination of three quarks; some of the combinations are given in Fig. 44-4*a*. With regard to baryon number, we see that any three quarks (each with $B = +\frac{1}{3}$) yield a proper baryon (with $B = +1$).

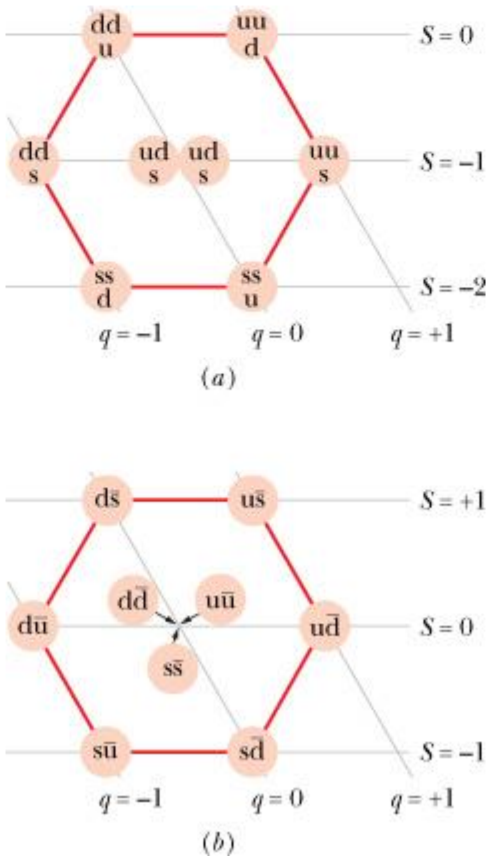


Figure 44-4

(a) The quark compositions of the eight spin-$\frac{1}{2}$ baryons plotted in Fig. 44-3a. (Although the two central baryons share the same quark structure, they are different particles. The sigma is an excited state of the lambda, decaying into the lambda by emission of a gamma-ray photon.) (b) The quark compositions of the nine spin-zero mesons plotted in Fig. 44-3b.

Charges also work out, as we can see from three examples. The proton has a quark composition of uud, and so its charge quantum number is

$$q(\mathrm{uud}) = \frac{2}{3} + \frac{2}{3} + \left(-\frac{1}{3}\right) = +1.$$

The neutron has a quark composition of udd, and its charge quantum number is therefore

$$q(\mathrm{udd}) = \frac{2}{3} + \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right) = 0.$$

The Σ^- (sigma-minus) particle has a quark composition of dds, and its charge quantum number is therefore

$$q(\mathrm{dds}) = -\frac{1}{3} + \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right) = -1.$$

The strangeness quantum numbers work out as well. You can check this by using Table 44-3 for the Σ^- strangeness number and Table 44-5 for the strangeness numbers of the dds quarks.

Note, however, that the mass of a proton, neutron, Σ^-, or any other baryon is *not* the sum of the masses of the constituent quarks. For example, the total mass of the three quarks in a proton is only 20 MeV/c^2, woefully less than the proton's mass of 938.3 MeV/c^2. Nearly all of the proton's mass is due to the internal energies of (1) the quark motion and (2) the fields that bind the quarks together (as discussed in Section 44-9). (Recall that mass is related to energy via Einstein's equation, which we can write as $m = E/c^2$.) Thus, because most of your mass is due to the protons and neutrons in your body, your mass (and therefore your weight on a bathroom scale) is primarily a measure of the energies of the quark motion and the quark-binding fields within you.

Quarks and Mesons

Mesons are quark – antiquark pairs; some of their compositions are given in Fig. 44-4*b*. The quark–antiquark model is consistent with the fact that mesons are not baryons; that is, mesons have a baryon number $B = 0$. The baryon number for a quark is $+\frac{1}{3}$ and for an antiquark is $-\frac{1}{3}$; thus, the combination of baryon numbers in a meson is zero.

Consider the meson π^+, which consists of an up quark u and an antidown quark $\bar{\mathrm{d}}$. We see from Table 44-5 that the charge quantum number of the up quark is $+\frac{2}{3}$ and that of the antidown quark is $+\frac{1}{3}$ (the sign is opposite that of the down quark).

This adds nicely to a charge quantum number of +1 for the π^+ meson; that is,

$$q(u\bar{d}) = \frac{2}{3} + \frac{1}{3} = +1.$$

All the charge and strangeness quantum numbers of Fig. 44-4*b* agree with those of Table 44-4 and Fig. 44-3*b*. Convince yourself that all possible up, down, and strange quark–antiquark combinations are used. Everything fits.

CHECKPOINT 3

Is a combination of a down quark (d) and an antiup quark ($\bar{u}$) called (a) a π° meson, (b) a proton, (c) a π^- meson, (d) a π^+ meson, or (e) a neutron?

Answer:

c

A New Look at Beta Decay

Let us see how beta decay appears from the quark point of view. In Eq. 42-24, we presented a typical example of this process:

$$^{32}\mathrm{P} \rightarrow {}^{32}\mathrm{S} + e^{-} + \nu.$$

After the neutron was discovered and Fermi had worked out his theory of beta decay, physicists came to view the fundamental beta-decay process as the changing of a neutron into a proton inside the nucleus, according to the scheme

$$\mathrm{n} \rightarrow \mathrm{p} + e^{-} + \bar{\nu}_{e},$$

in which the neutrino is identified more completely. Today we look deeper and see that a neutron (udd) can change into a proton (uud) by changing a down quark into an up quark. We now view the fundamental beta-decay process as

$$\mathrm{d} \rightarrow \mathrm{u} + e^{-} + \bar{\nu}_{e}.$$

Thus, as we come to know more and more about the fundamental nature of matter, we can examine familiar processes at deeper and deeper levels. We see too that the quark model not only helps us to understand the structure of particles but also clarifies their interactions.

Still More Quarks

There are other particles and other eightfold way patterns that we have not discussed. To account for them, it turns out that we need to postulate three more quarks, the *charm quark* c, the *top quark* t, and the *bottom quark* b. Thus, a total of six quarks exist, as listed in Table 44-5.

Note that three quarks are exceptionally massive, the most massive of them (top) being almost 190 times more massive than a proton. To generate particles that contain such quarks, with such large mass energies, we must go to higher and higher energies, which is the reason that these three quarks were not discovered earlier.

The first particle containing a charm quark to be observed was the *J*/*ψ* meson, whose quark structure is $c\bar{c}$. It was discovered simultaneously and independently in 1974 by groups headed by Samuel Ting at the Brookhaven National Laboratory and Burton Richter at Stanford University.

The top quark defied all efforts to generate it in the laboratory until 1995, when its existence was finally demonstrated in the Tevatron, a large particle accelerator at Fermilab. In this accelerator, protons and antiprotons, each with an energy of 0.9 TeV (= 9×10^{11}eV), are made to collide at the centers of two large particle detectors. In a very few cases, the colliding particles generate a top-antitop ($t\bar{t}$) quark pair, which *very* quickly decays into particles that can be detected and thus can be used to infer the existence of the top–antitop pair.

Look back for a moment at Table 44-5 (the quark family) and Table 44-2 (the lepton family) and notice the neat symmetry of these two "six-packs" of particles, each dividing naturally into three corresponding two-particle families. In terms of what we know today, the quarks and the leptons seem to be truly fundamental particles having no internal structure.

Quark composition of a xi-minus particle

The Ξ^- (xi-minus) particle is a baryon with a spin quantum number *s* of $\frac{1}{2}$, a charge quantum number *q* of -1, and a strangeness quantum number *S* of -2. Also, it does not contain a bottom quark. What combination of quarks makes up Ξ^- ?

Reasoning:

Because the Ξ^- is a baryon, it must consist of three quarks (not two as for a meson).

Let us next consider the strangeness *S* = -2 of the Ξ^-. Only the strange quark s and the antistrange quark $\bar{s}$ have nonzero values of strangeness (see Table 44-5). Further, because only the strange quark s has a *negative* value of strangeness, Ξ^- must contain that quark. In fact, for Ξ^- to have a strangeness of –2, it must contain two strange quarks.

To determine the third quark, call it x, we can consider the other known properties of Ξ^-. Its charge quantum number *q* is -1, and the charge quantum number *q* of each strange quark is $-\frac{1}{3}$. Thus, the third quark x must have a charge quantum number of $-\frac{1}{3}$, so that we can have

$$\begin{aligned} q(\Xi^-) &= q(\text{ssx}) \\ &= -\frac{1}{3} + \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right) = -1. \end{aligned}$$

Besides the strange quark, the only quarks with $q = -\frac{1}{3}$ are the down quark d and bottom quark b. Because the problem statement ruled out a bottom quark, the third quark must be a down quark. This conclusion is also consistent with the baryon quantum numbers:

$$\begin{aligned} B(\Xi^-) &= B(\text{ssd}) \\ &= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = +1. \end{aligned}$$

Thus, the quark composition of the Ξ^- particle is ssd.

44-9 The Basic Forces and Messenger Particles

We turn now from cataloging the particles to considering the forces between them.

The Electromagnetic Force

At the atomic level, we say that two electrons exert electromagnetic forces on each other according to Coulomb's law. At a deeper level, this interaction is described by a highly successful theory called **quantum electrodynamics** (QED). From this point of view, we say that each electron senses the presence of the other by exchanging photons with it.

We cannot detect these photons because they are emitted by one electron and absorbed by the other a very short time later. Because of their undetectable existence, we call them **virtual photons.** Because they communicate between the two interacting charged particles, we sometimes call these photons *messenger particles.*

If a stationary electron emits a photon and remains itself unchanged, energy is not conserved. The principle of conservation of energy is saved, however, by the uncertainty principle, written in the form

$$\Delta E \cdot \Delta t \approx \hbar. \qquad (44\text{-}18)$$

Here we interpret this relation to mean that you can "overdraw" an amount of energy ΔE, violating conservation of energy, *provided* you "return" it within an interval Δt given by $\hbar \Delta E$ so that the violation cannot be detected. The virtual photons do just that. When, say, electron *A* emits a virtual photon, the overdraw in energy is quickly set right when that electron receives a virtual photon from electron *B*, and the violation is hidden by the inherent uncertainty.

The Weak Force

A theory of the weak force, which acts on all particles, was developed by analogy with the theory of the electromagnetic force. The messenger particles that transmit the weak force between particles, however, are not (massless) photons but massive particles, identified by the symbols W and Z. The theory was so successful that it revealed the electromagnetic force and the weak force as being different aspects of a single **electroweak force.** This accomplishment is a logical extension of the work of Maxwell, who revealed the electric and magnetic forces as being different aspects of a single *electromagnetic* force.

The electroweak theory was specific in predicting the properties of the messenger particles. Their charges and masses, for example, were predicted to be

Particle	Charge	Mass
W	$\pm e$	80.4 GeV/c^2
Z	0	91.2 GeV/c^2

Recall that the proton mass is only 0.938 GeV/c^2; these are massive particles! The 1979 Nobel Prize in physics was awarded to Sheldon Glashow, Steven Weinberg, and Abdus Salam for their development of the electroweak theory. The theory was confirmed in 1983 by Carlo Rubbia and his group at CERN, and the 1984 Nobel Prize in physics went to Rubbia and Simon van der Meer for this brilliant experimental work.

Some notion of the complexity of particle physics in this day and age can be found by looking at an earlier particle physics experiment that led to the Nobel Prize in physics—the discovery of the neutron. This vitally important discovery was a "tabletop" experiment, employing particles emitted by naturally occurring radioactive materials as projectiles; it was reported in 1932 under the title "Possible Existence of a Neutron," the single author being James Chadwick.

The discovery of the W and Z messenger particles in 1983, by contrast, was carried out at a large particle accelerator, about 7 km in circumference and operating in the range of several hundred billion electron-volts. The principal particle detector alone weighed 20 MN. The experiment employed more than 130 physicists from 12 institutions in 8 countries, along with a large support staff.

The Strong Force

A theory of the strong force—that is, the force that acts between quarks to bind hadrons together—has also been developed. The messenger particles in this case are called **gluons** and, like the photon, they are predicted to be massless. The theory assumes that each "flavor" of quark comes in three varieties that, for convenience, have been labeled *red, yellow,* and *blue.* Thus, there are three up quarks, one of each color, and so on. The antiquarks also come in three colors, which we call *antired, antiyellow,* and *antiblue.* You must not think that quarks are actually colored, like tiny jelly beans. The names are labels of convenience, but (for once) they do have a certain formal justification, as you will see.

The force acting between quarks is called a **color force** and the underlying theory, by analogy with quantum electrodynamics (QED), is called **quantum chromodynamics** (QCD). Apparently, quarks can be assembled only in combinations that are *color-neutral.*

There are two ways to bring about color neutrality. In the theory of actual colors, red + yellow + blue yields white, which is color-neutral, and we use the same scheme in dealing with quarks. Thus we can assemble three quarks to form a baryon, provided one is a yellow quark, one is a red quark, and one is a blue quark. Antired + antiyellow + antiblue is also white, so that we can assemble three antiquarks (of the proper anticolors) to form an antibaryon. Finally, red + antired, or yellow + antiyellow, or blue + antiblue also yields white. Thus, we can assemble a quark–antiquark combination to form a meson. The color-neutral rule does not permit any other combination of quarks, and none are observed.

The color force not only acts to bind together quarks as baryons and mesons, but it also acts between such particles, in which case it has traditionally been called the strong force. Hence, not only does the color force bind together quarks to form protons and neutrons, but it also binds together the protons and neutrons to form nuclei.

Einstein's Dream

The unification of the fundamental forces of nature into a single force—which occupied Einstein's attention for much of his later life—is very much a current focus of research. We have seen that the weak force has been successfully combined with electromagnetism so that they may be jointly viewed as aspects of a single *electro-weak force.* Theories that attempt to add the strong force to this combination—called *grand unification theories* (GUTs)—are being pursued actively. Theories that seek to complete the job by adding gravity—sometimes called *theories of everything* (TOE)—are at an encouraging but speculative stage at this time.

44-10 A Pause for Reflection

Let us put what you have just learned in perspective. If all we are interested in is the structure of the world around us, we can get along nicely with the electron, the neutrino, the neutron, and the proton. As someone has said, we can operate "Spaceship Earth" quite well with just these particles. We can see a few of the more exotic particles by looking for them in the cosmic rays; however, to see most of them, we must build massive accelerators and look for them at great effort and expense.

The reason we must go to such effort is that—measured in energy terms—we live in a world of very low temperatures. Even at the center of the Sun, the value of kT is only about 1 keV. To produce the exotic particles, we must be able to accelerate protons or electrons to energies in the GeV and TeV range and higher.

Once upon a time the temperature everywhere *was* high enough to provide such energies. That time of extremely high temperatures occurred in the **big bang** beginning of the universe, when the universe (and both space and time) came into existence. Thus, one reason scientists study particles at high energies is to understand what the universe was like just after it began.

As we shall discuss shortly, *all* of space within the universe was initially tiny in extent, and the temperature of the particles within that space was incredibly high. With time, however, the universe expanded and cooled to lower temperatures, eventually to the size and temperature we see today.

Actually, the phrase "we see today" is complicated: When we look out into space, we are actually looking back in time because the light from the stars and galaxies has taken a long time to reach us. The most distant objects that we can detect are **quasars** (*qua*sistell*ar* objects), which are the extremely bright cores of galaxies that are as much as 13×10^9 ly from us. Each such core contains a gigantic black hole; as material (gas and even stars) is pulled into one of those black holes, the material heats up and radiates a tremendous amount of light, enough for us to detect in spite of the huge distance. We therefore "see" a quasar not as it looks today but rather as it once was, when that light began its journey to us billions of years ago.

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